**Unit 6: Investigation 4 (2 Days)**

**Transformations of Trigonometric Functions**

**Common Core State Standards**

F.IF.7eGraph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F.BF.3 Identify the effect on the graph of replacing *f*(*x*) by *f*(*x*) + *k*, *kf*(*x*), *f*(*kx*), and *f*(*x* + *k*) for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

**Overview**

Students will apply what they have learned about the transformations of functions in previous units to the trigonometric functions: sine, cosine and tangent. They will learn about the amplitude, period and midline of trigonometric functions as they are associated with the vertical and horizontal stretches, and vertical shifts. STEM intending students should learn about the phase shift for trigonometric functions as horizontal shifts.

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Students will be able to find the three other representations of a trigonometric function given one of the following representations: algebraic formula, graph, tables, verbal description.

**Assessment Strategies: How Will They Show What They Know?**

* **Exit Slip 6.4.1** provides a verbal description of the vertical height of a rider on a Ferris wheel as a function of time and has students fill in a table of values, sketch the graph and write an algebraic formula for the function. They must identify the amplitude, period and midline of the trigonometric function.
* **Exit Slip 6.4.2** provides students with an algebraic formula for sine, cosine or tangent function and students need to apply their knowledge of transformations of functions to sketch a detailed graph of the function. They must identify the amplitude, period and midline of the trigonometric function.
* **Exit Slip 6.4.3** gives students a graph of a sine, cosine or tangent function and students need to identify the amplitude, period and midline of the trigonometric function. They also need write an equation for the graph.
* **Journal Prompt 1** Students aregiven the general form of a sine function,

f(x) = a(sin(bx)) + d

and are asked to explain how each of the three parameters a, b and d affect the graph of y = sin(x). Students need to identify which parameter is associated with the amplitude, period and midline of the trigonometric function. STEM intending students should include the 4th parameter ‘c’ for horizontal shift, also known as phase shift in a trigonometric function: f(x) = a (sin(b(x - c))) + d

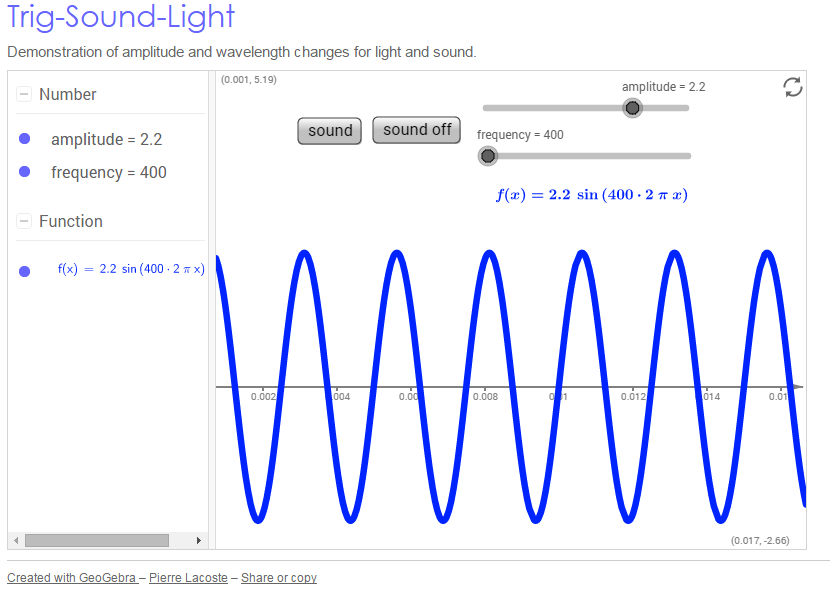
* **Activity 6.4.1 Move It!Trig/** **Activity 6.4.1(+) Move It! Trig.** As was done in Unit 1, Investigation 2 where students were first introduced to the effect of each parameter in the function *a*f(*b*(x - *c*)) + *d* = y, students through exploration determine how the parameter *d* affects the graph of the sine, cosine or tangent functions.Activity 6.4.1(+) explores horizontal as well as vertical shifts, hence *c* and *d*.
* **Activity 6.4.1 b Move It! Trig with Geogebra** **/Activity 6.4.1b (+) Move It! Trig with Geogebra.** If students can use Geogebra applets, give them the Activity 6.4.1bbefore, after or in place of Activity 6.4.1 Move It! Trig. Note these activities do not provide the table skill building that Activities 6.4.1 and 6.4.1+ do.
* **Activity 6.4.2 Stretch it**! **Trig/ Activity 6.4.2 b Stretch it**! **Trig with Geogebra**

is similar to the transformation of function activities in Unit 2 on Quadratics. Students experiment with multiplying parameters a(f(x)) and f(bx) for the trig functions and observing the vertical and horizontal compressions and stretches. The b version uses Geogebra and does not have the table building activity in it that 6.4.2 does.

* **Activity 6.4.3 Graphs from Equations and Equations from Graphs.** Students will obtain trigonometric equations from graphs and graph trigonometric equations.
* **Activity 6.4.4** **Interpreting Real World Sinusoidal Models** provides graphs of Ferris wheel heights, water wheel heights, ocean water heights and pendulum swings vs time. Students will need to obtain equations from the graphs of these applied phenomena and interpret in context the meaning of the amplitude and period for each model.

**Launch Notes**

If you have access to a computer, a really effective launch is to use one of the many available applets that play a pure tone (musical note) while simultaneously showing the graph and equation of the sine function of the number of vibrations as a function of time. One slider changes the period of the graph and the pitch of the note. Another slider changes the amplitude of the graph and the volume of the note. Seeing the transformations of the graph while hearing the changes in sound interests many students. One such applet is included in the materials for this investigation, and is also available on the internet. It is the Geogebra file “material-95819sound.ggb” called Trig-Sound\_Light by Pierre Lacoste. <https://tube.geogebra.org/m/95819> Here is a screen shot:



Ask the students “what would happen if I move the slider this way….” , then play the sound to test their predictions. Or ask how to move the slider to achieve a certain sound.

Another applet created by Dr. Robert Decker called “TrigBeats” is included here or you may ago to his website <http://uhaweb.hartford.edu/rdecker/> in the middle right of the page is the link [State of CT algebra curriculum project](http://uhaweb.hartford.edu/rdecker/algebra/algebra.html) in tiny letters. You will then see “Adding trig functions and beats; hear and see it. Click on [beats](http://uhaweb.hartford.edu/rdecker/algebra/trigBeats.exe)”. The applet is called “trig beats” and has a help menu on the top row of links.

If you have a tablet, you can also download any number of free apps for piano keyboards.

Take a few moments to describe what sound is, and why it is modeled by a sine wave. Sound is produced when something- (clap your hands)- causes air molecules around the sound source to vibrate. These vibrating air molecules vibrate the molecules next to them and so on down the line till the air molecules next to your ear are vibrated. You can illustrate this compression and rarefication of air molecules by lining up 6 or 7 students shoulder to shoulder facing the class. Start an end student swaying, by having her step on first the left then the right foot. The first student represents the air molecules that are set in motion by a noise – you could clap your hand to start the first person swaying. As the first person sways left to right, she bumps the person next to her and he starts moving back and forth. The swaying moves along down the line of people. The person at the end of the line represents the set of air molecules near your ear; the air molecules that start your ear drum vibrating. The vibrations picked up by the ‘hairs’ in the inner ear are translated to a neurological signal that is experienced by your brain as sound.

Because of the way sound is processed by our brains, the way we perceive a sound might not be exactly in accordance to the mathematical changes we make. For example, doubling the amplitude does not result in our perception that the volume was doubled. We will hear an increase in volume, but we can’t say exactly how much of an increase. Pitch affects our perception of volume, as well. A singer can sing the high notes more softly than she sings the low notes, knowing that the audience will perceive the notes to be the same volume. (Try singing more softly next time you have to reach for that high note.)

It is important to see that graph of the sine waves do not show the path of a molecule of air, but rather, the compression and rarefication of air that ripples through adjacent air molecules. The dependent variable for a graph of a sine wave that represents sound is air pressure, and the independent variable is time. You can model an air molecule moving back and forth – getting compressed and released - by having a motion detector plot the graph of the last person in line swaying back and forth. Notice the similarity with the pendulum swing.

The pitch of a sound is measured in hertz – vibrations per second, or frequency. The frequency of the Concert A that orchestras’ tune to is 440 Hz. That means the air molecules vibrate 440 times in one second, and can be written as y = sin(440x). Write the frequency as a fraction or rate: 440 vibrations /second, and then ask students to write and interpret the reciprocal of frequency. Guide them to understand that 1 second for 440 vibrations, or 1/440th of a second for 1 vibration is the period of the function. That is to say, the reciprocal of the frequency is the period. Seeing this helps some students understand why an increase in the coefficient of x inside a function – i.e. f(bx) – causes a horizontal shrink, that is a decrease in period. If a vibration happens twice as fast, the frequency is doubled, and the period will be half as long. The period of sin(2x) is half the period of sin(x).

See if there are any more questions about sound, write them on the board, and encourage students to find answer to their questions. If students do the sound activities in Investigation 5 and in the performance task they may find some answers. They can also talk to the physics teacher. Don’t feel you have to be able to answer all the questions. Perhaps you can pick a few to assign to students for an independent project. Encourage students to take courses in the physics of sound and acoustics to learn more.

Briefly review the transformation of functions that students saw starting in Units 1 and 2. Show the equation y = *a*(f(*b*x)) + *d* on the board. Show a particular quadratic (e.g., f(x) = -3(2x)2 +5) in this form. Have them articulate that multiplication of a parameter results in a stretch or squeeze, addition of a parameter results in a shift. Inside parameters effect horizontal changes, and outside parameters effect vertical changes.

**Teaching Strategies**

**Activity 6.4.1 Move It!Trig/** **Activity 6.4.1(+) Move It! Trig.**  As was done in Unit 1 Investigation 2 where students were first introduced to the effect of each parameter in the function a(f(b(x - c))+ d = y, challenge students to find how each parameter a, b, and d affects the graph of the sine, cosine or tangent functions. Remember that only STEM intending students need concern themselves with ‘c’, the phase shift. In Activity 6.4.1 Move It! Trig, students start with y = sin(x) + k and observe the effect on the parent function y = sin(x) for different values of k. The second activity 6.4.2 investigates the effect of parameters ‘a’ and ‘b’. The third activity 6.4.3 pulls it all together. Note the activities using Geogebra do not include the table building activities that the nonGeogerbra platform does.

**Activity 6.4.1 b Move It! Trig with Geogebra.** If students can use Geogebra applets, give them the **Activity 6.4.1b** before, after or in place of **Activity 6.4.1 Move It! Trig**. This **Activity 6.4.1b** gives directions for using the file Activity\_6\_4\_1\_b\_GeoGebra\_Code\_06\_22\_15 that is an applet with sliders for changing the parameter k on a graph of a function such as y = sin(x) + k. If you prefer to search for a different Geogebra applet, an internet search for “Transformations of Trig Functions with Geogebra” yields many options. Activity 6.4.1(+) and Activity 6.4.1b (+) are identical to the 6.4.1 and 6.4.1b, except for the phase shift lessons that are added for STEM intending students.

**Activity 6.4.2 Stretch it**! **Trig Activity/ 6.4.2 b Stretch it**! **Trig with Geogebra** is similar to the transformation of function activities in Unit 2 on Quadratics. Students experiment with multiplying parameters a(f(x)) and f(bx) for the trig functions and observing the vertical and horizontal compressions and stretches. As in Activity 6.4.1, there is an activity sheet for Geogebra applets called **Activity 6.4.2 b Stretch it**! **Trig with Geogebra** that uses the file Activity\_6\_4\_2\_b\_GeoGebra\_Code\_06\_22\_15

**Activity 6.4.3 Graphs from Equations and Equations from Graphs.** After students know the effect of each parameter on the basic graphs of the trigonometric functions, and before you distribute Activity 6.4.3, have a class discussion to synthesize the translations and transformations for sine, cosine and tangent functions. For example, you can work through f(x) = 3sin(.25x) - 5, g(x) = -3sin(.25x) - 5, h((x)= (-1/2)cos(4x) + 1 and j(x) = -2tan(3x) + 1. Summarize by writing directions for graphing the trigonometric functions.

A student might explain how to graph trigonometric functions like this: for sine and cosine, start with sketching the midline, then sketch horizontal lines where the maximum and minimum values occur (you can call this the ‘envelope’ within which the function lives). Next determine the phase shift (it should be 0, except for STEM intending students). Determine the period by dividing 360° or 2π by parameter b. Mark the points that are the start and the end of the period on the horizontal axis; partition that interval into fourths; plot points at the midline and the extrema, for at least one period. If the function is a positive sine curve, then the graph should “start” at the midline, make a hill, then a valley all within one period. If a negative sine, the graph starts at the midline and decreases to make a valley, then the hill. A positive cosine curve will start at the maximum point, reach the minimum half way along in the period, and end the period at the maximum again.

Graphs of the form y = a∙tan(bx) + d are made by plotting the “starting point” at (0, d), identifying the period as 180°/b or π/b, drawing the vertical asymptotes that edge one period-- start and end, and plotting the points that correspond to (π/2, 1) and (-π/2, 1) from the parent function, that is the “quarter points. Sketch a few periods. A vertical stretch will be difficult to indicate unless students label the points that are a quarter way through the period. Vertical stretches for tangent functions need not be emphasized.

Still in class discussion, present just the graph of a function such as y = −5sin(1/2 x)+3, and have students determine the algebraic formula from the graph. From the graph, students should determine the amplitude, period and vertical shift in order to write an equation for the graph of a trigonometric function. Note that many equations are possible from any one graph if students are using a phase shift.

STEM intending student will work with phase shifts that are not zero, so have a class discussion that develops students’ understanding of phase shifts and periodicity. For example, you could have each group graph equations that are equivalent but have different algebraic representations, such as f(x) = sin(x), g(x) = sin(x - 2π) , h(x) = -sin(x - π) and j(x) = sin(x + 2π). Simultaneously have the groups show their graph to the class. They will discover that the graphs are identical. Now ask students to explain why the functions are equivalent using transformations and periodicity.

Have students explain how to determine the phase shift. A student might say to first pick out either a single period of the sine curve (the wave starts in the middle) or a cosine curve (the wave starts and stops at the maximum value). Call the x coordinate of the “start” the parameter ‘c’ that is the phase shift or horizontal shift. Students will realize that infinitely many algebraic equations can describe one trigonometric graph depending on where one chooses to “start” the period.

Once you have gone over a few examples of graphing a trigonometric function from an equation and finding an equation from a graph, students should be prepared to do **Activity 6.4.3** on their own in groups or for homework.

**Activity 6.4.4** provides graphs of Ferris wheel heights, water wheel heights, ocean water heights and pendulum swings vs time. Students will need to obtain equations from the graphs of these applied phenomena and interpret in context the meaning of amplitudes and periods of these models. Later in the unit students will use data in these contexts and sinusoidal regression and need to interpret the amplitude, period, maximums and minimums in context.

**Group Activity**

**Activity 6.4.1 Discovering Transformations of Trig Functions** is a discovery lesson, and should be done in groups.

**Group Activity**

**Activity 6.4.3 Graphs from Equations and Equations from Graphs** lends itself well to the jigsaw technique in group work. Have students work through problems 1-4 in a heterogeneous group of size 5. Form new homogenous groups from the existing groups. Try to make the regrouping seem random by giving the least able student in each group a certain color or denomination playing card. Then give the next level student a different color or card denomination. Continue until five large homogenous groups are formed. Assign 1 or 2 problems from exercises # 5-21 to each group according to level of difficulty. You can break the large groups into smaller groups maintaining homogeneity. Each group will work 1 or 2 problems that match their ability level. You may choose which of the problems you assign to each group. Once each student is proficient in their problem(s) they return to their original heterogeneous groups. Each student will teach his original group how to work his problem(s). Students can then do the remaining 10 or 15 problems in class or for homework.

**Differentiated Instruction (For Learners Needing More Help)**

Since the most difficult parameter to understand is the horizontal stretch, the parameter b in y = sin(bx), provide extra practice for students needing more help. Some of the exercises should be about drawing ‘b’ periods of a sine curve in the interval from 0 to 360 degrees, and marking the start and end of each new period. Students should note that they had to divide the usual period into ‘b’ new periods. Because students will have to divide 360 by b to determine the points on the x axis where the new periods begin and end, they can then make the connection that the period is 360 degrees/b or 2π/b.

**Differentiated Instruction (Enrichment)**

Students can learn about phase shifts (horizontal shifts) of the trigonometric functions. Students needing additional challenge can examine f(x) = a sin((bx - c) + d and determine the phase shift.

**Journal Prompt 1**

Given the general form of a sine function, f(x) = a sin((b(x - c)) + d, ask students to explain how each of the four parameters a, b, c and d affect the graph of y = sin(x). Identify which parameter is associated with the amplitude, period, phase shift and midline of the trigonometric function. Students should include in their varied answers that amplitude = │a│ and a is the vertical stretch/compression factor, d is the vertical shift and the y = d is the equation of the midline, period = 2π/b and b is the horizontal stretch/compression factor and Stem students should discuss the role of c.

**Closure Notes**

By the end of Unit 6, Investigation 3, students will be able to graph the functions y = sin(x), y = cos(x) and y = tan(x) and apply the transformations to these new functions. Given a function in symbolic form or described in words, students can sketch its graph and identify the amplitude, period, turning points, and midline. STEM intending students will also be able to state the horizontal or phase shift. Given a graph of a sinusoidal or tangent function, students will be able to identify the same features of the graph and will be able to write an equation that models the function.

To wrap up the lesson, consider returning to the discussion of musical sound, and play a concert “A” that is the ‘A’ above middle ‘C’. Concert “A” is the note orchestras tune to, so you could show a short video of an orchestra tuning or ask for a class volunteer to play or sing the note. If you use tablets, you can have students download one of many free apps for a virtual piano keyboard. The equation for concert ‘A’ is y = sin(440x), where the range variable is air pressure and the domain variable is seconds. We say that the note ‘A’ is 440 vibrations per second, or 440 hertz. Can students identify the period (It is 1/440th of a second)? The note one octave below is half the frequency (220 hertz), and the period is1/220th of a second. Next ask how we would amend the equation for the concert ‘A’ to get the note one octave below. Sketch the graph of concert A and the note one octave below (y = sin(220x)).

Next ask student how to alter the equation of concert A to increase its volume (Answer: increase the amplitude). Have students write a possible equation and sketch the graph for a louder ‘A’. (One possible answer is y = 2sin(x).)

**Vocabulary**

Inside and outside changes

Parameter

Transformations of functions

**Resources and Materials**

**Activities 6.4.1, 6.4.2, 6.4.3 6.4.4 should be done in this investigation**

Activity 6.4.1 Move It! Trig

Activity 6.4.1(+) Move It! Trig

Activity 6.4.1b Move It! Trig with Geogebra (directions)

Activity 6.4.1b(+) Move It! Trig with Geogebra (directions)



Activity 6.4.2 Stretch It! Trig

Activity 6.4.2b Stretch It! Trig with Geogebra (directions)



Activity 6.4.3 Graphs from Equations and Equations from Graphs

Activity 6.4.4 Interpreting Real World Sinusoidal Models

Uncooked spaghetti (2-3 strands per student)- linguini will not roll.

Double faced tape (or scotch tape)

Graphing Calculator

Geogebra

Function Graphers with sliders for changing parameters while the corresponding musical tone is played such as can be found in Geogebra (search for

