**Unit 5: Exponential and Logarithm Families**

**UNIT OVERVIEW**

30 days (May take longer if you must go back to Unit 7 in Algebra 1)

The Algebra 2 unit on exponential and logarithmic functions builds upon the concepts explored in Algebra 1 Unit 7 and Algebra 2 Unit 1. Instructors are encouraged to revisit the core concepts of those units, especially Algebra I Unit 7, as substantial time may have elapsed since that material was covered or it may have been omitted completely. The investigation overviews will recommend specific investigations from the Algebra 1 curriculum and activities within those investigations.

Building upon the concept of a geometric sequence, characterized by a common quotient, students develop the continuous exponential function and in particular the natural base *e.* Examination of the graph of will lead students to conclusions regarding the possible values of *b* and *a*, as well as the domain and range of the function, and that the function is in fact an invertible function. All but the latter were investigated in Algebra 1 Unit 7. The definition of the appropriate inverse function g(*x*) = logb *x* and the assumption of continuity will permit students to develop methods for solving exponential and logarithmic equations.

Initially, the focus of this unit should be on the graphs of the logarithmic and exponential functions, and answering fundamental questions regarding them. If for example, is assumed to be continuous, students will readily recognize that the solution to is the x-value of the common point of intersection of the graphs of and . Or they can observe that they are looking for the input that produces an output of 8. A richer development results from considering the intersection of and a horizontal line whose height is not an integral power of 2, such as, . Under the assumption of continuity for it is graphically clear that an intersection exists and that therefore has a solution. Moreover, encouraging the student to describe this solution *in words,* without concern for a specific value, leads to the statement that *“x is the exponent to which 2 is raised to produce 14”* and the translation to . Building along these lines leads to the definition of the inverse function, . Remind them that as in Unit 1 Investigation 6, when f and f-1 exist, f(c) = d is equivalent to f-1(d) = c and stating this definition in terms of the exponential and the new logarithm function f(x) = bx is equivalent to f-1(x) = logb x. Emphasize that a logarithm is an exponent.

The development of the logarithmic function as the inverse of the exponential function and the understanding that a logarithm is an exponent will assist students in discovering the rules for logarithms. For example, students can be asked to find

, and compare this result to . With a sufficient number of examples and some prompting to look for a relationship between the numbers *m*, *n* and *z(=mn*) the “Product Rule” can be established and the “Power Rule” is seen to be a special case by noting:

, *t* a Real number. Of course *m* and *n* must both be positive. The quotient rule follows easily from these results.

Applications and the multiple real world situations where logarithms are needed are investigated: sound and the decibel, earthquakes and their scales, measuring PH, brightness of stars, continuous compounding and other financial applications provide a rich and interesting source that avoids the student asking, “Where am I ever going to use this?”

The interrelationship between multiple representations verbal (words), numerical or tabular, graphical, and symbolic (equations) continues to be emphasized. Students will continue to be asked to defend their statements, to formulate definitions and to use precise mathematical language and appropriate symbols.

The Exponential/Logarithm unit consists of 7 investigations of which 1, 2, 4, 5 and 6 form the essential core needed for further study. Instructors pressed for time should ensure that these investigations are covered thoroughly. Investigations 3 and 7 are valuable extensions for all citizens in today’s world but some of the activities can be omitted in the interest of time. The investigation overviews will provide further information.

**Investigation 1**. Through use of a paper folding activity students examine the basic exponential function f(x) = 2x. It is assumed students have done some work in unit 7 of Algebra 1 and therefor have already met this function. The exponential function focused on given x, find y. But the question is now posed: given y, find x and the logarithmic function is defined. Basic rules of logarithms are discovered and used. Work with base 2 and base 10 are emphasized.

**Investigation 2** extends the concepts of the first investigation by defining *e*, the base of the natural logarithm function using the notion of compound interest. In activities, students discover as well as . Students model exponential growth problems and solve , by using the natural logarithm function, and use technology where needed to find an approximate solution.

**Investigation 3** compares linear to logarithmic scales. When the range of data values is very large, say from 1 to scaling the numbers is a natural technique to use in order to make graphing reasonable. Using a linear scale, if it is too small the larger numbers cannot fit and if a larger linear scale is used the smaller numbers become indistinguishable. It is not that the numbers are too big or too small, the problem arises when the numbers vary too greatly in size. Logarithmic scales are used in measuring the magnitude of earthquakes, measuring the pH levels of acidity or alkalinity in a substance, and measuring sound in decibels. Activities for the former logarithmic scales and others are provided.

**Investigation 4** begins with a problem that needs a model with an exponential function that has an exponent other than 1*x*, setting the stage to examine transformations on the graph of and the roles of the parameters *a*, *b*, and *c* in f(x) =*abc*x. Students examined the roles of a and *b* in Unit 7 of Algebra 1. Exponential functions of the form are rewritten as and the solutions to are expressed in terms of logarithms.

**Investigation 5** will look at curve fitting using technology as students did in Unit 5 in Algebra 1 and in Unit 2 of Algebra 2, but this time using the exponential regression or logarithmic regression equation available with technology. The need to make a scatter plot first becomes more relevant to students now that they have several available potential regression equation choices. Students will also pass an exponential through two points. An extension will be to plot *ln y* by *x* and if linear to then transform the linear equation using *x* and *ln y* as the variables to obtain the exponential equation in x and y.

**Investigation 6** defines a geometric series, in general, and computes a finite geometric sum. Finite arithmetic series and their sums were examined in unit 3. Sam’s story from Activity 5.1.1 is continued, the spread of rumors, worms and computer viruses are examined and Sierpinski’s carpet is explored. Students met Sierpinski’s Triangle in Algebra 1.

**Investigation 7** explores personal financial mathematics with activities on car payments where the Future Value of an annuity is developed and extended to use for longer term savings, mortgages and amortization tables, and the formula for the monthly payment of a loan and comparing financing options are examined.

**Essential Questions**

* Why do some mathematical models have limitations when used to model a real world situation?
* When you are deciding upon a mathematical model to use, what factors must you consider?
* How do you know that the exponential function is invertible?
* What characterizes logarithmic growth?
* What characterizes exponential growth and decay?
* What are real world models of exponential and logarithmic growth and decay?
* What are the limitations of exponential growth models?
* How can one differentiate an exponential model from a linear model given a real world data set?

**Enduring Understandings**

* When comparing an exponential model with a linear model, the question is not *if* the exponential model will generate very large or very small inputs, but rather *when*.
* Compounding whether in a savings account or on a loan is powerful.
* Many functions have inverse functions that can undo them.

**Unit Understandings**

* When a function is one-to one we can undo it and define an inverse function.
* The logarithmic form can always be rewritten in exponential form and vice versa.
* A logarithm is an exponent.
* The laws of exponents make the laws of logarithms sensible and easy to remember.
* With real data, sometimes deciding whether data is linear or non-linear is more complex than just looking at a graph, differences ( yn – yn-1), or an r-value; it is important to examine differences that are approximately the same more carefully to see if there is a pattern of increasing or decreasing values that, because the pattern is exponential, soon begin to produce outputs with extremely large or small values.
* An informal limiting process permits us obtain a continuous compounding formula from the finite compounding formula and define the number *e*.
* The domain of a logarithmic family member guarantees that the argument of the function is nonnegative.
* A logarithmic scale is needed when the numbers we need to graph vary greatly in size.
* The logarithmic and exponential families are rich in applications to the real world.

**Unit Contents**

Investigation 1: (3 days)Logarithmic Functions- Inverse of Exponential Functions

Investigation 2: (3 days)Natural logarithms and base *e*

Investigation 3: (4 days) Logarithmic Scales

Investigation 4: (4 days)Parameters of Exponential Functions

Investigation 5: (4 days) Curve Fitting with Exponential and Logarithmic Functions

Investigation 6: (3 days) Geometric Series

Investigation 7: (4 days) Financial Mathematics

Performance Task: (1 day for presentations)

Review and Mid Unit Assessment (2 day)

Review for Unit Test (1 day)

End of Unit Test (1 day)

**Common Core Standards**

*Mathematical Practices #1 and #3* *describe a classroom environment that encourages thinking mathematically and are critical for quality teaching and learning. Practices in bold are to be emphasized in the unit.*

1. Make sense of problems and persevere in solving them.

2. Reason abstractly and quantitatively.

3. **Construct viable arguments and critique the reasoning of others.**

4. **Model with mathematics.**

5. Use appropriate tools strategically.

6. **Attend to precision**.

7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning.

**Common Core State Standards**

A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

A-SSE-1b Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret as a product of *P* and a factor not depending on *P.*

A-SSE-4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. For example, calculate mortgage payments.

A-CED-1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*

A-CED-2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F-IF. 7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F-IF-8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

F.BF.1 Write a function that describes a relationship between two quantities.

F-BF-3 Identify the effect on the graph of replacing by and for specific values of (both positive and negative);

find the values of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

F.BF.4a Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. For example, f(x) = 2(x3) or f(x) = (x + 1)/(x – 1) for x ≠ 1.

F.BF.4b (+) Verify by composition that one function is the inverse of another.

F.BF.4c (+) Read values of an inverse function from a graph or a table, given that the graph has an inverse.

F-BF-5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents**.**

F-LE-4 For exponential models, express as a logarithm the solution to where *a, c* and *d* are numbers and the base *b* is 2, 10 or *e;* evaluate the logarithm using technology.

F-LE-5 Interpret the parameters of an exponential function in terms of a context.

A.REI.11b Explain why the x-coordinates of the points where the graphs of the equations y = f(x) and y = g(x) intersect are the solutions of the equations f(x) = g(x); find the solutions approximately, e.g. using technology to graph the functions, make tables of values, or find successive approximations. Include cases where f(x) and/or g(x) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

**Assessment Strategies:**

**Performance Task**

Students are asked to create a mural or a timeline on a single sheet of paper that shows the relative positions of events long ago as well as the events of very recent times. There is a linear timeline on the long sidewalk called The Walk Through Time at the Rocky Hill Dinosaur State Park that connects the parking lot to the building. Students are asked to put the same information on a piece of paper, but to use a logarithmic scale to show the long time periods (how long ago dinosaurs began) as well as to provide enough detail to show the shorter time periods (how long ago mammals appeared, how long ago humans appeared, how long ago recorded history began…). A power point slide presentation has been provided in the teacher resources so that students who have not visited the park can see the long linear sidewalk. So students will be charged with creating a mural or brochure or handout that reflects the information on the sidewalk – that shows the timeline for how long ago was the start of life to the present. Here is the Dinosaur state park (Rocky Hill) home page. <http://www.dinosaurstatepark.org/>

You can offer an alternative. Or if you do not want everyone working on the same mural suggestions follow.

For example, if you go to the Smithsonian Institute website shown here and below <http://humanorigins.si.edu/evidence/human-evolution-timeline-interactive>, there is an interactive timeline for zooming in at short time periods (mammals) and zooming out at longer time periods (age of the dinosaurs). Students could be charged with, “How could you get all this information on a sheet of paper?”

Other alternatives are suggested in the resources below.

**Resources for Performance Task:**

Here is the Dinosaur state park (Rocky Hill) home page. <http://www.dinosaurstatepark.org/>

And power point of the photos of the walk in Rocky Hill Dinosaur State Park is found in the teacher resources folder.

Smithsonian suggestion:

<http://humanorigins.si.edu/evidence/human-evolution-timeline-interactive>

You could tell students that they have to “correct” the famous mural at the Yale Peabody Museum: Nice pictures, but is the timeline visually accurate? i.e. How accurately does it depict the relative length of the Jurassic period to, say, the cretaceous period ?

<http://peabody.yale.edu/sites/default/files/images/store/reptileposter.jpg.crop_display.jpg>

Time Line for either the Big Bang Timeline and click on Universe by the Numbers on the left side once you use the link: <http://www.physicsoftheuniverse.com/topics_bigbang_timeline.html>

A graph from Wikipedia Chronology of the Universe that can be adapted and needs a logarithmic scale:

<http://en.wikipedia.org/wiki/Chronology_of_the_universe#cite_note-BICEP2-2014-2>

**Other Evidence (Formative and Summative Assessments)**

* Exit slips
* Class work
* Homework assignments
* Math journals
* Unit 5 Mid-unit assessment
* Unit 5 End of Unit assessment

**Vocabulary**

|  |  |
| --- | --- |
| Base ***e*** | Natural Logarithm |
| Change of Base Rule | One-to-one function |
| Common quotient | Power Rule |
| Compound interest | Product Rule |
| Continuous compounding | Quotient Rule |
| Exponential function | Strictly decreasing |
| Exponential function base ***e*** | Strictly increasing |
| Finite Sum |  |
| Geometric Series |  |
| Growth factor |  |
| Inverse function |  |
| Invertible |  |
| Logarithmic function |  |