**Activity 8.4.2 Area of a Circle from Regular Polygons**

In his book, *Journey through Genius* (1991), William Dunham described Archimedes of Syracuse (287 – 212 BC) as “a much revered, somewhat eccentric genius who dominated the mathematical landscape of the classical world.” (p.84) Among his multiple remarkable accomplishments Archimedes created a method for estimating the area of any circle, which, at the same time, created a surprisingly accurate estimate for what we now call π (pi.) We will use some of his ideas but with tools he did not have.

1. The ancients knew that for all circles, the ratio of the circumference to the diameter is constant. Today we call that ratio π.
2. Using what you know about similarity, explain why this constant ratio makes sense.
3. What is the constant ratio for any square between its perimeter and side length?
4. What is the constant ratio for any square between its perimeter and diagonal length?
5. Show how to derive a formula for the circumference of a circle in terms of its radius from the constant ratio $\frac{c}{d}$= π

In this activity we are going to reconstruct Archimedes method using trigonometry, a tool not available to him.

1. Archimedes’ idea was that the area of a circle could be estimated by finding areas of egular polygons inscribed in and circumscribed around a fixed circle.

  

1. What does it mean for a polygon to be inscribed in a circle?
2. What does it mean for a polygon to be circumscribed around a circle?
3. Suppose you could find the areas of both an inscribed and a circumscribed regular polygon. How would that help you estimate the area of the circle?

In Unit 5 you learned that you can circumscribe a circle about every triangle, but not every quadrilateral. So in general, not all polygons have circumcircles. However ,we can show that every regular polygon has a circumcircle. For a proof, see question 11 below. We’ll call the center of the circumscribed circle the **center** of the regular polygon.

3. Let’s see how to find areas of regular polygons.

1. One way to think about the area of a regular polygon is to decompose it into triangles with vertices at the center. Think about the case of the regular pentagon.

1. How many triangles are formed?
2. What types of triangles are they? How do you know?
3. How do you know the triangles are congruent?
4. The **apothem** of a regular polygon is the segment from the center of the regular polygon to the midpoint of any side. How do you know the apothem is also perpendicular to the side?
5. Draw the apothem in one of the triangles. How do you know the apothem is the same regardless of which side you draw it to?

1. Let *a* represent the length of the apothem. Let *s* represent the length of one side of the regular pentagon. Using *a* and *s*, write a formula for the area of each triangle.
2. Use your formula in (g). What is a formula for the area of the entire regular pentagon?
3. One general formula for the area of a regular polygon with apothem *a* is A = $\frac{1}{2}$ *pa* where *p* represents the perimeter of the polygon. Explain why this formula makes sense. Hint: Use your formula in (h) for the regular pentagon and see how you can generalize it.
4. Let’s begin to use Archimedes’ method. Consider the equilateral triangle inscribed in the circle. Let the radius of the circle be 1.

1. Decompose the equilateral triangle into isosceles triangles. Find the angle measures in the isosceles triangle. Explain your reasoning.
2. Draw an apothem to one side of the equilateral triangle forming two right triangles. What are the measures of the acute angles one of these triangles?
3. Use the cosine ratio to find an expression for the height of the isosceles triangle.

1. Use the sine ratio to find an expression for the length of the base of each isosceles triangle. Hint: Use the right triangle again.
2. Find the area of each isosceles triangle.
3. Find the area of the equilateral triangle.
4. Repeat the work from question 4 with a regular pentagon inscribed in a circle. Again, let the radius of the circle again be 1.
5. Decompose the regular pentagon into isosceles triangles. Find the angle measures in the isosceles triangle.
6. Draw an apothem to one side of the regular pentagon forming two right triangles. What are the measures of the acute angles one of these triangles?

1. Use the cosine ratio to find the height of the isosceles triangle. Hint: See 4(c).

1. Use the sine ratio to find the base of each isosceles triangle.
2. Find the area of each isosceles triangle.
3. Find the area of the regular pentagon.
4. Let’s generalize to find a formula for the area of any regular polygon inscribed in a circle of radius 1. Assume the regular polygon has *n* sides. 

1. How many isosceles triangles will you get when you decompose the *n*-gon by drawing radii to each vertex?
2. What is the measure of central angle that intercepts one side of the regular polygon?
3. Draw an apothem to one side of the regular polygon forming two right triangles with the radii and side. Let $θ$ be the measure formed by the apothem and the radius. Find $θ$ in terms of *n.*$ $
4. Use the cosine ratio to find the height of each isosceles triangle.
5. Use the sine ratio to find the base of each isosceles triangle.
6. Find the area of each isosceles triangle.
7. Find a formula for the area of a regular *n*-gon inscribed in a circle of radius 1.
8. Check your formula for a square. Inscribe a square in a circle of radius 1.
9. Sketch your square inscribed in a circle of radius 1.

1. Find the area of the square without trigonometry.
2. Find the area of the square using the trigonometry formula you found in 6(g).

1. Your results should agree. If not, rework your reasoning.
2. Archimedes also circumscribed regular polygons around a circle. Our goal now is to find an expression for any regular *n*-gon circumscribed around a circle of radius 1. The diagram shows a regular heptagon (7 sides) but think about it as a general case. 
3. Subdivide the regular *n*-gon into isosceles triangles.
4. Write an expression for the measure of each central angle (for example $∠OHB$ in the figure at the right).
5. Draw a segment from the center of the circle to one of the points of tangency ($\overbar{OH}$ in the figure above). Explain why this forms right triangles (e.g. ∆*OHB* in the figure).
6. What role does the radius of 1 play in relation to the isosceles triangles?

1. Study the small right triangle (half the isosceles triangle). Use the tangent ratio to write an expression for the base of the triangle.
2. Find an expression for the area of the regular *n*-gon.
3. Complete the table.

Note that (a) and (b) refer to inscribed regular polygons. (Use 6(g). Parts (c) and (d) refer to circumscribed polygons (Use 8(f)).

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| --- | --- | --- | --- | --- |
| **Regular****Polygon** | **a. Formula for Inscribed Polygon** | **b. Area for Inscribed Polygon** | **c. Formula for Circumscribed Polygon** | **d. Area for Circumscribed Polygon** |
| Equilateral triangle*n* = 3 | 3 sin60° cos60° | 1.299 | 3 tan 60° | 5.196 |
| Regular quadrilateral *n* = 4 |  |  |  |  |
| Regular Pentagon*n* = 5 |  |  |  |  |
| Regular Hexagon*n* = 6 |  |  |  |  |
| Regular Heptagon*n* = 7 |  |  |  |  |
| Regular Octagon*n* = 8 |  |  |  |  |
| Regular Nonagon*n* = 9 |  |  |  |  |
| Regular Decagon*n* = 10 |  |  |  |  |
| Regular Dodecagon*n*= 12 |  |  |  |  |
|  |  |  |  |  |

1. a. What happens to the areas of the two polygons (inscribed and circumscribed)
 as *n* gets larger?
2. Choose an even larger value of *n* and fill in one more row of the table. What values to the inside and outside areas appear to be approaching? Why?

11. Prove that every regular polygon has a circumscribed circle.

1. Start with a regular polygon. For our purposes we will use an octagon, *ABCDEFGH.*
2. Draw the perpendicular bisectors of two consecutive sides ($\overbar{AB }$and $\overbar{BC}$). Let *P* be the point where they intersect. Let *I* and *J* be the midpoints of $\overbar{AB}$ and $\overbar{BC}$ respectively.
Show that *PA = PB = PC*.
3. Show that $\vec{BP}$ bisects $∠CBA$.

We have now shown that *P* is the same distance from vertices *A*, *B*, and *C* and that $\vec{BP}$ bisects $∠CBA. $To complete the proof we need to show that *P* is the same distance from the other vertices of the regular polygon.



1. Show that $∠PCB$ $≅∠PCD$.

1. Show that *PC = PD.*

1. Explain why steps (e) and (f) could be repeated for all the remaining vertices of the regular polygon, no matter how many sides there are.
2. Explain why we now know why the circle with center *P* and radius = *PA* will pass through all the vertices of the regular polygon.