**Unit 6: Investigation 3 (2 Days)**

**Graphs of Trigonometric Functions**

**Common Core State Standards**

F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F.TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

**Overview**

Now that students have defined trigonometric functions with a domain of all real numbers, they can work on two other representations of the function: ordered pairs (tables of values) and graphs. Use the vertical and horizontal position of a person riding a Ferris wheel to introduce the graphs and vocabulary of the sine and cosine functions. By plotting the known x or y coordinates on a unit circle from the wrapping function, students will obtain graphs of y=sinx, y=cosx, .Students will see that the tangent is the slope of the terminal ray of angle t, (or the y coordinate over the x coordinate of P(t)) . A “Sine Tracer” animation or applet will help students see how the wrapping function on the unit circle traces out a sinusoidal

**Assessment Activities:**

**Evidence of Success: What Will Students Be Able to Do?**

* Make a table of values for the sine, cosine and tangent functions of a real number.
* Recognize the range of the sine function is the y coordinate of the wrapping function and the vertical position of the point P(t) on the unit circle, the range of the cosine function is the x coordinate of the range of the wrapping function and the horizontal position of the point, and that the range of the tangent is the ratio of the y to the x coordinates of the range of the wrapping function and the slope of the terminal ray of angle t, for any P(t).
* Graph the sine, cosine and tangent functions of a real number showing period, midline and amplitude
* For each graph, label the x axis in both radians and degrees

**Assessment Strategies: How Will They Show What They Know?**

* **Exit Slip 6.3.1** Complete a table of values and graph 2 periods of the rider’s vertical distance from ground as a function of time for a Ferris wheel has a radius of 6 meters, takes 12 minutes to complete one rotation and is mounted 1 meter above the ground. Assume the Ferris wheel begins to rotate when the rider is at the 3 o’clock position and the wheel rotates counterclockwise. Sketch a picture of the Ferris wheel and its dimensions.
* **Journal entry 6.3.1** Sketch one period each of the 3 trigonometric functions, clearly identifying the x intercepts, the period and the amplitude. Write a way that you can remember these graphs. (one example of student writing could be: “The basic sine function starts at (0,0) makes a hill and then a valley. The cosine starts on the y axis at the top of the wave, goes to the bottom and ends at 2π at the top of the wave again. The tangent function is like a snake, is increasing thru the origin and has vertical asymptotes at π/2, 3π/2, 5π/2 which is where the terminal ray of angle t is vertical.”)
* **Activity 6.3.1A** **Investigating Spaghetti Curves** has students use raw spaghetti to measure the length of the vertical distance from a fly’s height to the ground using a unit circle. The spaghetti pieces are transferred to a coordinate system, with the endpoints of the pieces forming a sine curve.
* **Activity 6.3.1A Part 2** has students use an applet that not only reinforces the hand graph they made with spaghetti but permits them to study the effect changing the measure of the radius of a circlehas on the equation of the sine function.
* **Activity 6.3.1B High up on the Wheel** that has information from 4 or 5 different Ferris wheel rides from which to make tables of values, plot the points and sketch the graph of height of the rider as a function of time. For each wheel, students will draw a sketch of the Ferris wheel, and identify the amplitude, the period and the midline of the graph of height as a function of time
* **Activity 6.3.2 Graph the Big Three** students will learn the graphs of the parent functions y = sin(x), y = cos(x), and y = tan(x).

**Launch Notes**

Draw a unit circle on the board with the center diameter. Ask students to imagine a fly crawling along the unit circle from 0 to $\frac{π}{2}$ radians, from$ \frac{π}{2} $ to π, from π to $\frac{3π}{2}$ radians, and from $\frac{3π}{2}$ to 2π radians. As the fly walks along the circular path, its distance above or below the ground changes. (Distance below the ground is negative.)

If we were to plot the fly’s distance from the ground with respect to distance traveled around the unit circle, what would be the shape of the graph?

**Teaching Strategies**

Tell students they will use a technique to generate the graph of the fly’s distance from the x axis using spaghetti. Give students **Activity 6.3.1A** and ask them:

* As the fly moves in a counterclockwise direction from $\frac{π}{2} $ to π radians, will the spaghetti pieces get longer? Why or why not?
* As the flymoves in a counterclockwise direction from π to 2π radians, will the spaghetti pieces get longer? Why or why not?

Have them complete the activity. You can then have students make a cosine graph using spaghetti and horizontal distances, or you can move to **part 2** of this activity in which students complete a table of values and plot the points to form the graph of y = sin(x).

After they have graphed the sin curve (and cos curve) open the virtual spaghetti applet at the following web address: <http://tinyurl.com/spagh2>.

1. Click on the **Insert Spaghetti** button to insert a few “spaghetti” starting at 0. Tell them this is exactly what they did.
2. Move point *P* along the unit circle to have student answer the following questions.
3. How does the length of the spaghetti piece (represented as a gold segment) change as point *P* is dragged about the unit circle?
4. What is the maximum length that a spaghetti piece may attain? When or where is this maximum reached?
5. What is the minimum length of any spaghetti piece? When or where is this minimum reached?
6. Are spaghetti lengths unique for a particular angle measure? In other words, is it possible for two different angle measures to have the same corresponding spaghetti length? Why or why not?
7. Place P at 0 then select Trace at the top of the screen and move point *P* along the unit circle
8. Check the Trace box and drag *P* around the circle. What do you observe?
9. The algebraic representation for the spaghetti graph generated by a circle of radius 1 is  *y* = 1sin(*x*).
10. Ask students: How can we change *y* = 1 • sin *x* to make the function match the graph generated by a circle with radius 2?
11. Check the Radius box and drag the point to 2 units. Move point *P* along the unit circle. Ask students how does the radius measure change the graph?
12. Next, enter 3sin(x) in the **Input bar** at the bottom of the applet screen. Discuss how the shape repeats itself and discuss period of the curve.

Introduce the word “amplitude” and help student associate amplitude with the radius of the circle – the distance from the center to the maximum point, or to the minimum point, half the distance from the maximum to the minimum point.

Have students fill in the table on **Activity 6.3.1A Part 2**

Before you distribute **Activities 6.3.1B High Up on the Wheel**. Remind students about the Ferris wheel discussed in Investigation 1. Ask students can the Ferris wheel be at the same position as the unit circle that the fly was walking on? Why? How far up should we bring the unit circle? Using a paper unit circle, move it so the bottom of the circle is touching the ground, asking them is this a good height. Discuss with class what would be a good height and why?

Now you can give the students some facts about the Ferris wheel.

The original Ferris wheel was built by George Ferris for the Columbian Exposition of 1893. It was much larger and slower than its modern counterparts. The diameter of the Ferris wheel was 250 feet and contained 36 cars, each of which held 60 people; it made one revolution every 10 minutes. The wheel was mounted so that its lowest point was 2 feet above the ground.

You can ask students to imagine riding the wheel and that the wheel begins to rotate when the seat you are in is at the 3 o’clock position. Find the distance you are above ground as a function of time.

Fill in a table of values for t = 0, 2, 4, 6, 8....16 minutes. Then have them graph the points on the coordinate plane.

Ask students if your height changes at a constant rate as you ride the Ferris wheel. When is height increasing most rapidly (at the 3 o’clock position); decreasing most rapidly (at the 9 o’clock position)? When does the height change most slowly (at the top and bottom of the wheel)? Note that the maximum value of the graph occurs when the rider is at the top of the wheel. The minimum value of the function occurs when the rider is at the bottom of the wheel. Ask students what is the period of the function, and how many periods you have graphed. Note that the period is the same whether you measure peak to peak, or minimum to minimum. If we measure from where the graph crosses the midline, to the next point of intersection with the midline, half a period will have passed. Note that the rider will be at the same height twice in each rotation: once on the way up and then on the way down (except, of course, for the maximum and minimum heights.) Identify the midline and associate the midline with the average height of the rider above ground– i.e. half the time the rider is above the midline and half the time the rider is below the midline. If the students would benefit from another example, have them graph the height function of another Ferris Wheel with, say, a different radius.

**Differentiated Instruction**

Have students find a video of a Ferris wheel for themselves, or even go to the amusement park and take a video. Ask them to graph vertical height of a rider as a function of time without telling them any other information. They can find the period with a timer. Using stop action, students can measure the size of the wheel in the photo. To determine a scale factor, they can estimate from a nearby known measure – e.g., the height of a man who measures 3 cm in the photo is 6 feet tall in real life.

Distribute **Activity 6.3.1B High up on the Wheel** that has information from 4 or 5 different Ferris wheel rides from which to make tables of values, plots of the points and sketches of the graph of height of the rider as a function of time. For each wheel, students will draw a sketch of the Ferris wheel, and identify the amplitude, the period and the midline of the graph of height as a function of time. If a group is done early, you can have the students design their own Ferris wheel to work with. If there is not enough time in class, students can finish this activity for homework.

**Group Activity – Activity 6.3.1B. High Up on the Wheel** Students can work in pairs or groups of three to make the graphs, table of values, and respond to the questions asked regarding the equation of the midline, period, amplitude for the Ferris wheels in the activity.

 **Activity 6.3.1B High up on the Wheel** presents information about the radius of the wheel, the time it takes a wheel to make one rotation, and the height at which the wheel is mounted, including a wheel that is half underwater so that its midline is at ground level (y = 0). Students will discover what happens to the period of the graph when a wheel goes twice as fast or three times more slowly. For now, always have the wheel start at the 3 o’clock position and go counterclockwise. Students should gain practice dividing the wheel in fourths and focusing on what the coordinates for time and height are at the start, quarter way along the ride, half way and three quarters the way around corresponding to the 3, 12, 9 and 6 o’clock positions on the wheel. Do not try to make connections with the transformations of graphs. The main idea is to learn the vocabulary and features of the sine function in the context of a Ferris wheel ride. In Investigation 5, students will thoroughly explore the transformations of trigonometric functions. **Exit Slip 6.3.1** is appropriate at this time.

**Differentiated Instruction**

As an alternative to using **Activity Sheet 6.3.1B High up on the Wheel** in group try using a jigsaw form of group work. Start the students in their heterogeneous groups, but prearrange for students of like ability to be grouped together, by, for example, distributing different color coded cards to students in each group and pulling all students of one color to join their new ability-based group. Separate out the problems in **Activity Sheet 6.3.1B High up on the Wheel** by level of difficulty so you can assign two or three of the problems to each group according to ability. Once each group is done with their problems, they rejoin their original group and each student teaches their problems to the original heterogeneous group.

In **Activity 6.3.2 Graph the Big Three** students will learn the graphs of the parent functions y = sin(x), y = cos(x), and y = tan(x) . Tell the students that we will return to the unit circle so we can learn to graph the basic trigonometric functions, because we defined the basic functions in terms of the unit circle. Encourage students to see that riding on the Ferris wheel is analogous to seeing the terminal point of an arc t go from 0 to 2π around the unit circle.

You might want to reinforce the idea that simple, elementary, brute force methods such as graphing functions by plotting points are effective and informative, though not necessarily elegant. Have student make a table of values for y = sin(x) with x = 0, π/6, π/4, π/3, π/2, 2π/3, 3π/4, 5π/6, π, 3π/2, 2π. By using symmetry, students will not need to find the values for all the special angles on the unit circle, just those in the first and second quadrant should suffice. Focus on filling in the exact value for sin(t) first, so that the ordered pair is clear from the table. There is space in the table to write decimal approximations (to two decimal places) and the degree equivalent for each radian measure. Have students plot a **rough draft** of the graph by plotting these points by hand on graph paper. Have them observe that from 0 to π/6, the y coordinate increases by .5, but the increase is much less for, say, the interval from π/3 to π/2. Ask questions to help students see that as the rate of change decreases, the graph is concave down (and definitely not linear). Note the amplitude, the extrema, the midline, the x intercepts and the period. Next, have students draw a final copy of the graph of y = sin(x) with exact radian and degree measures on the x axis at every integer multiple of π/2 on the activity sheet at the base of the table of values. The graph should extend from -2π to 2π.

Depending on what your students need, at some point in this activity, you may want to introduce a “Sine Tracer” which is a computer or calculator animated graphic that shows a point moving around a circle while simultaneously tracing out a sine wave. If you do a search for Sine Tracer you will find several computer applications to choose from. You can also search the geogebra files to find examples such as **Graph of Sine from Wrapping Function (Tau)** file name on geogebra: material-135299.ggb by Doug Kuhlman. The link is <https://tube.geogebra.org/m/1352199> .

Another option is for students to make their own Sine Tracer on the graphing calculator. You will find directions attached to this investigation. Directions are essentially the same for any TI83 or 84 calculator. Use simultaneous graphing in parametric mode in radians, use the graph that is a bubble with a tail for the first function that is the circle., x1 = cos(t), y1 = sin(t). The second function is the sine wave: x2 = t, y2 = sin(t). Use the enter key to pause and resume the graphing action. The ‘on’ key will abort the graph. The ‘enter’ key will stop the graph, and pressing ‘enter’ again will restart where you stopped. To restart the sine tracer, alter the window ever so slightly – e.g. change the t-step, then hit ‘graph’. To slow down the trace, decrease the t-step. You can search the internet for directions for a TI 84 Sine Tracer. One such option is attached at the end of this overview.

In **Activity 6.3.2 Graph the Big Three** students will follow the same directions but this time will draw a cosine graph. If you haven’t already done so, you can have students use spaghetti strips to measure the horizontal distance, then make a table of values to plot. Have students amend the instructions for Sine Tracer in order to make a Cosine Tracer. Ask questions to help students see that the cosine function is a horizontal shift of the sine function. Challenge students to use what they learned about translations of functions (f(x - c)) to write an equivalent expression in terms of sin(x) for the graph of the cosine function. You may choose to use the Ferris Wheel information and graph a function of the rider’s horizontal distance from center as a function of time so that the students can see an instance where cosine is associated with horizontal and sine is associated with vertical aspects of circular motion.

Lastly, have students plot points and graph the function for y = tan(x), and create a Tangent Tracer. Be sure to draw the connection between the slope of the angle t and the y coordinates of the tangent function. Of particular interest is the period of tangent – note that the slope for the terminal ray of angles in quadrants 1 and 3 are identical and slopes of angles in quadrants 2 and 4 are identical. Note also that as the angle moves along the angles from 89°, say, to 91° the slope changes from being steep and positive, to being steep and negative. The 90° angle is the point where the tangent function jumps from positive values of large magnitude (y approaches ∞) across the vertical asymptote to negative values of large magnitude (y approaches -∞). Advise students that, unfortunately, some graphing calculators connect these two points and mistakenly draw a line that is nearly vertical that incorrectly crosses the asymptote. They can go to the mode key and use DOT instead of CONNECTED Mode to correct the mistaken impression that the vertical or nearly vertical lines are part of the graph. So we are smarter than a graphing calculator!

By the end of **Activity 6.3.2 Graph the Big Three,** students will have rough drafts of y = sin(t), y = cos(t), and y = tan(t), on a separate sheets of standard graph paper. They will have the table of values completed on the activity sheet along with a clean, neat graph of each trigonometric function sketched at the bottom of the page with each table. They will complete short answer and fill-in-the-blank questions about the trigonometric functions and their graphs.

**Journal entry 6.3.1** Sketch one period each of the 3 trigonometric functions, clearly identifying the x intercepts, the period and the amplitude. Create a pneumonic that you can remember these graphs. (One example of student writing could be: “The basic sine function starts at (0,0) makes a hill and then a valley. The cosine starts on the y axis at the top of the wave, goes to the bottom and ends at 2π at the top of the wave again. The tangent function is like a snake, is increasing through the origin and has vertical asymptotes at π/2, 3π/2, 5π/2 which is where the terminal ray of angle t is vertical.”)

**Closure**

We have pursued the sine function from right triangle to unit circle to waves. Ask students to sketch an illustration of these three manifestations – a 30-60-90 degree triangle, a unit circle and a graph of y = sin(x). Indicate how or where each shows the sin(30°) (or the equivalent sin(π/6). Ask them to indicate sin(-5π/6) in order to show that the right triangle does not allow us to find the sine of, say, -150° or -5π/6, since these numbers are not in the domain of right triangle trigonometric functions. However, the circular definition of trigonometry has every real number in its domain, so we can now find the sine of any real number. We can now use trigonometric functions to model many periodic functions in the real world even though there is no triangle. Have students begin to think about all sorts of periodic behaviors they see in the world: around and around, up and down, back and forth.

**Vocabulary**

Amplitude

Maximum

Midline

Minimum

Odd and even symmetry

Period

**Resources and Materials**

**Activities 6.3.1A, Activity 6.3.1A Part 2**, **6.3.1B and 6.3.2 should be completed in this investigation**

Activity 6.3.1A Investigating Spaghetti Curves

Activity 6.3.1 Part 2 Virtual Spaghetti

Activity 6.3.1B High Up on the Wheel

Activity 6.3.2 Graph the Big Three

Illustrations of Ferris Wheels

Graph Paper

Graphing Calculator

Two sided tape

Software like Geogebra for running a sine tracer file such as “**Graph of Sine from Wrapping Function (Tau)”** a geogebra file named material-135299.ggb by Doug Kuhlmann. <https://tube.geogebra.org/m/1352199> (A file is attached as “material-1352199 Sine Tracer from W(t))” OR you can download the file from the internet.)

**Sine Tracer TI84 Activity\_6\_3\_2 Directions** to distribute for creating a sine tracer on the TI 83/83 using parametric graphing mode. (Document attached to end of this overview. Also available in the Teacher Notes)

**Sine Tracer directions for the TI 83/84**



From <http://mathbits.com/MathBits/TISection/Trig/unitcircle.htm>

Link to the Geogebra file “Sine Tracer using Wrapping Function” :



A search for “sine tracer geogebra” or “sine graph from unit circle geogebra” will give you several options to choose from.

Screen Shot from Sine Tracer by Doug Kuhlmann on Geogebra

