**Unit 6: Investigation 1 (4 Days)**

**The Unit Circle and Radian Measure**

**Common Core State Standards**

F.TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

**Overview**

In this investigation students will learn that the standard position for a central angle on a unit circle has its initial ray coincide with the positive x-axis, its vertex at the origin and for a positive angle, the terminal ray sweeps counterclockwise around the circumference of the circle; will define a radian; will note the equivalence of arc length on a unit circle and the radian measure of the central angle that subtends the arc (On the unit circle a central angle that measure two radians subtends an arc whose length is 2 radius units.); will learn that the unit circle is the circle centered at the origin that has a radius of 1 unit and algebraically, the unit circle is the set of points that satisfy the equation x2 + y2 = 1;will understand that the special angles in radians and degrees on the unit circle correspond to quarters, sixths, eights and twelfths of the circle and will be able to convert between degree and radian measure of an angle.

Students will focus especially on arc lengths and central angles for positive integer multiples of π, π /2, π /3, π /4, and π /6 .

Investigation 1 provides the background information needed to formulate the circular definition of sine, cosine and tangent that will occur in Investigation 2.

*Note to teachers: In this unit, we will talk about the measure of the central angle in degrees or radians and the length of an arc in some unit of linear measure. We will not talk about the arc measure in degrees.*

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Define radian measure in words and by illustration.
* Find the arc length or the radian and degree measure of the central angle that the arc subtends by using the equivalence of arc length with the central angle measure on a unit circle: “s = t.”
* Divide a unit circle into halves, fourths, sixths, eighths and twelfths. Label the central angles in degrees and radians.
* Convert between radian and degree measure.
* For any real number t, sketch an angle of measure t in standard position.
* Determine whether two angles are co-terminal and given an angle in standard position, determine the measures (positive and negative) of angles that are co-terminal with the given angle.
* Define a unit circle as a circle whose radius is 1 unit in measure. State the algebraic formula for a unit circle. Be able to graph the unit circle, find points on the unit circle, and determine whether or not a point lies on the circle.

**Assessment Strategies: How Will They Show What They Know?**

* **Exit slip 6.1.1** asks students to sketch the angles π/3, 5π/3, π/4, 5π/4 and/or π/6, and 5π/6 in standard position on the unit circle.
* **Exit Slip 6.1.2** has students graph the following angles in standard position, and determine which two are co-terminal: 21π/4, -7π/6 and 870°.
* **Exit Slip 6.1.3** asks students to sketch the arc that has a length of $\frac{π}{6}$ in standard position on the unit circle. Students label the coordinates of the terminal point of this arc ($\frac{\sqrt{3}}{2},\frac{1}{2}$), confirm that the point with coordinates ( $\frac{\sqrt{3}}{2},\frac{1}{2}$ ) is on the unit circle by checking to see if the coordinates satisfy the equation x2 + y2 = 1, sketch the arcs that measure $\frac{5π}{6}, \frac{7π}{6} and-\frac{π}{6} $ and determine the coordinates for the terminal points of these arcs.
* **Journal Prompt 1** has students state in words what a radian is and draw an illustration of a “radian.”
* **Journal Prompt** **2** asks students to explain how to convert radians to degrees and degrees to radians, to explain how to decide which factor to use, to use the conversion factor to find approximately how many degrees in one radian, and to sketch an arc and a central angle that measures 1 radian.
* **Activity 6.1.1 Angle Measure Can Be Any Number of Degrees** has students think about the domain of the sinusoidal function h(t) - height of a rider on a Ferris wheel as a function of time - compared to the domain of the right triangle sine function.
* **Activity 6.1.2 What is a Radian?** has students learn what a radian is by constructing a circle with a compass on tracing or other translucent paper. Each student should have a different size circle.
* **Activity 6.1.3 Learn More About Radians and Arc Lengths** continues to explore the concepts from the previous activity, and there is additional practice with arc lengths and radians, that leads students to realize that a central angle measure of 2π radians is equal to the central angle measure of 360°.
* **Activity 6.1.4 Arc Length as Direct** **Variation** provides a review of direct variation; i.e. y = kx or s = r𝜽: arc length is proportional to the size of the central angle measured in radians.
* **Activity 6.1.5 Radian Measure of Central Angles as the Ratio** $\frac{s}{r}$guides students to formulate an alternative description of the radian measure of a central angle as the ratio of the arc length subtended by the angle to the length of the radius.
* **Activity 6.1.6 The Unit Circle** has students work with the unit circle algebraically, graphically and verbally. They will find points on the circle, and use the symmetry of the circle to label corresponding points. Students should be able to do the activity on their own with little explanation from you.
* **Activity 6.1.7 Special Angles and Arc Lengths** not only introduces students to the radian measure for the angles in the special triangles they learned in geometry – the 30-60-90 degree and the isosceles right$ $triangles - but also provides students with practice and drill with fractions.
* **Activity 6.1.8 and 6.1.8a Convert Between Degrees and Radians** has students learn the conversion factor (180°/π radians) for radians to degrees and also from degrees to radians (π radians/180°). Students will perform a variety of conversions and sketch the angles in standard position. **Activity 6.1.8a** is identical to **6.1.8** except that the first page of Activity **6.1.8a** has more extensive instruction and practice for converting various units of measure.

**Launch Notes**

**Periodic Behavior: Round and Round, Up and Down**

Begin the class by showing a promotional video or photo of the Ferris wheel ride at a local amusement park or show a famous wheel. You could even give an assignment to students the night before to find one fact about Ferris wheels to share in class.

Share some facts about the first Ferris wheel built by George Ferris for the Columbian Exposition of 1893. It was much larger and slower than its modern counterparts. The diameter of the Ferris wheel was 250 feet and contained 36 cars, each of which held 60 people; it made one revolution every 10 minutes. The wheel was mounted so that its lowest point was 2 feet above the ground.

Tell the students to draw a sketch of the original Ferris wheel showing the dimensions of the wheel including the radius, the lowest point on the ride and the highest point on the ride.

Have the students imagine that they are riding the wheel and that the wheel begins to rotate counterclockwise when the seat they are in is at the 3 o’clock position. Engage the students in conversation about how high off the ground you are at certain times as you ride the wheel h(t). Where are you gaining height most rapidly? What is it like at the top? Does it feel as if you are suspended at the top for a relatively long time? Where on the circular path of the wheel are you losing height fastest? At the point you descend fairly quickly, do you feel your stomach in your throat or your hair blowing in the breeze as you descend? Try to get the students to imagine what it is like to be riding that wheel and feeling the up and down movement as the wheel goes round and round.

Ask students to fill in a table for a function h(t) that shows the height of the rider above ground at t = 0, 2.5, 5, 7.5, 10…20 minutes since the ride started. Then have them graph the points on the coordinate plane. Ask how we should connect the points. Does height change at a constant rate as you ride the Ferris wheel? If so, then we should connect the points with line segments. (No, change in height over time is not a constant.)

Connect the points with a sinusoidal curve. Tell students that we will be studying special kinds of periodic functions called the Trigonometric Functions. Quickly review the basic information about right triangle trigonometry by asking the students what they remember about trigonometry from their previous classes. If they can’t remember, have them research it for homework. Three points to elicit:

* Trigonometry is based on the proportionality of sides in similar figures (Corresponding angles are equal in measure, sides are proportional).
* If an acute angle in one right triangle has the same measure as an acute angle in another right triangle, then by angle-angle, the triangles are similar - the acute angles and the right angles are congruent.
* In similar right triangles, the ratio of two sides in relation to a given acute angle is constant, regardless of the lengths of sides. The ratio of the length of the leg opposite an angle to the length of the hypotenuse is called the sine of the angle, for example.

Get students wondering about what right triangle trigonometry from geometry has to do with the Ferris wheel and the circular trigonometry we are about to study.

Discuss one or two other examples of sinusoidal-like functions, such as those below, and have the students sketch a very rough graph for several periods of each function. Be sure that the students label the horizontal and vertical axes with a reasonable domain and range. Make clear to students that these examples are not exactly sinusoidal functions, but they can be roughly modeled by a sine or cosine curve. Introduce the concept of ‘periodic functions’ to students, - i.e. a function that has repeating values on a regular interval. You can point out that an easy way to determine the period for a function with a graph that looks like a wave is to find the distance along the horizontal axis from peak to peak. Also show that the period can also be measured between any two corresponding points on the sinusoidal wave: such as minimum point to minimum point. Ask students how we could use the points that intersect the midline of the wave. (Successive points at the midline define half a period. OR successive points at the midline as the function is increasing defines one period.)

For whichever couple of examples you choose, have students identify the independent variable, dependent variable and period in each example:

* the percentage of area of the moon that is lighted as a function of the day of the month. (domain: number of days since start of the year, range: percentage of moon lit, period: approximately 1 month)
* the height of the water at an ocean pier as a function of hours elapsed since a given time (domain: number of hours elapsed, range: height of water, period: approximately 12 hours)
* average daily temperature in Connecticut as a function of month. (domain: number of days since January 1, range: average daily temperature, period:1 year)
* The height of a bungee jumper as a function of time since she leapt off the platform. The jumper rebounds every 3seconds. This particular bungee bounce lasts for 2 minutes. You can model this with a slinky bobbing up and down. Note the damping effect in this example. (domain: time elapsed since jump, range: height above ground, period: 3 seconds)

Consider using a motion detector to graph the sine wave that models

* a swinging pendulum’s horizontal distance from the motion detector as a function of time. (Provided the angle formed by the pendulum with a vertical line is small, this function will be approximated by a sinusoidal function.)
* the distance that the bottom of an oscillating slinky is from the floor as a function of time

(Note that the harmonic motion for both the slinky and the pendulum is damped, and the relative maximum points on the graph are modeled by an exponential decay function.)

* Have a person walk away from then toward, then away from and then toward the motion detector. The distance the person is from the motion detector as a function of time as he is walking back and forth in front of the motion detector is approximately sinusoidal.

The goal of the Launch is for students to have some models of sinusoidal functions in mind as they start this unit, to identify the independent and dependent variables, identify a reasonable domain and range, and to estimate the period of the function.

**Teaching Strategies**

**Activity 6.1.1 Angles Measure Can Be Any Number of Degrees** starts students thinking about the domain of the sinusoidal function h(t) - height of a rider on the Ferris wheel as a function of time - compared to the domain of the right triangle sine function. If the Ferris wheel operator gives the riders some extra free revolutions, or if we measure time in seconds, then we would want the domain of the vertical height function to be able to be a very large number. How can we expand the domain of sine function to include numbers larger than 90 degrees? Students learn that angles can be any real number of degrees if we measure the rotation of an angle centered at a circle. They learn about central angles in standard position and co-terminal angles. Be sure they understand that one revolution is 360°; two is 720°, etc. The last page can serve as a short homework assignment.

In **Activity 6.1.2 What is a Radian?** students learn what a radian is by constructing a circle with a compass on tracing or other translucent paper. Each student should have a different size circle. Using the radius of a circle as the unit of measure to make a special tape measure out of string that is marked with a felt tip pen, students measure an arc 1 radius long starting at the point (1,0) on the circumference of the circle, moving counterclockwise. They draw the central angle subtended by the arc of length 1 radius, and thus learn the definition of radian as the measure of the central angle subtended by an arc that is the length of one radius. When students place their papers on top of each other, lining up the centers of the circles, they will see vividly that an angle that measures 1 radian is the same size regardless of the length of the radius. To save time, or if you don’t have compasses, you may make and copy different sized circles for the students to use. If you are going to skip Activity 6.1.3 “Learn More About Radians and Arc Length”, then be sure students understand that an arc of length 2π units subtends an angle measuring 2π radians, and 2π radians equals 360 degrees.

**Activity 6.1.3 Learn More About Radians and Arc Lengths** continues to explore the concepts from the previous activity, and there is additional practice with arc lengths and radians, that leads students to realize that a central angle measure of 2π radians is equal to the central angle measure of 360°. From this equivalence students derive equivalences between radian and degree measure. (180° = π radians, 90° = π/2 radians, 60° = π/3 radians, 45° = π/4 radians, 30° = π/6 radians). If you are pressed for time, you can eliminate this activity, and summarize the equivalences on the board during class, or you could have students start the first page in class, check their understanding and then finish the other pages for homework.

**Activity 6.1.4 Arc Length as Direct** **Variation** provides a review of direct variation; i.e. y = *k*x or s = r𝜽: arc length is proportional to the size of the central angle measured in radians. The constant of proportionality is the length of the radius of the circle. This activity is not essential, or can be assigned for homework since direct variation was reviewed in unit 4 and studied in Algebra 1.

**Activity 6.1.5 Radian Measure of Central Angles as the Ratio** $\frac{s}{r}$guides students to formulate an alternative definition of the radian measure of a central angle as the ratio of the arc length subtended by the angle to the length of the radius. **Be sure to do this** **activity**. By the end of the activity, we see that radian measure of an angle can be thought of as a real number, thus allowing us to use trigonometric functions of real numbers should we want to model periodic behaviors with various independent real variables - not just angles measured in degrees.

**Differentiated Instruction**

Some students may not be comfortable with creating a unit of measure that is not traditional like the inch or centimeter. Ask what non-traditional or non-standard units of measure we may use in our daily lives. (Ex: when mixing juice concentrate with water, we may use “juice can” as the unit of measure and add three juice cans of water to one juice can of concentrate.) You could have students use the first joint of their finger as a linear unit of measure, measure something with it, and then have students research the history of the ‘inch’ as a unit of measure.

**Journal** **Prompt** **1** Have students state in words what a radian is and have students draw an illustration of an angle that measures 1 radian. Students might respond that a central angle of one radian has an intercepted arc with length 1 radius unit.

In the previous activities, students learned that it is the unit circle that allows us to identify the length of an arc with the radian measure of the angle it subtends. **Activity 6.1.6 The Unit Circle** has students work with the unit circle algebraically, graphically and verbally. They will find points on the circle, and use the symmetry of the circle to label corresponding points. Students should be able to do the activity on their own with little explanation from you. If you give it as a homework assignment, first show the students one example of finding the positive and negative x coordinates given y = $\frac{2}{\sqrt{3}}$. Then graph these points ($\pm \frac{1}{2},\frac{2}{\sqrt{3}}$) and the 2 other corresponding points ($\pm \frac{1}{2},-\frac{2}{\sqrt{3}}$) on the unit circle using symmetry. Next ask if the point (3/4, 3/4) is on the circle. (Answer: No, because substituting the values in for x and y in the formula for the circle gives a false statement.) With this short presentation, students could do **Activity 6.1.6 The Unit Circle** for homework.

(An historical note to share with the students is that Rene Descartes (1596-1650) is called the Father of Modern Mathematics, because he brought algebra and geometry together to what we call analytic geometry or coordinate geometry. By using algebraic equations and their coordinate points on a plane, for example, we can describe and work with geometrical objects like points, lines and circles. The name “Cartesian Coordinates” is in honor of Descartes, though Pierre de Fermat (1601-1665) may deserve at least equal credit for this discovery because he had the idea first. The marriage of geometry and algebra made the invention of calculus possible in 1665 by Isaac Newton and at roughly the same time by Gottfried Leibniz. Rene Descartes is also the Father of Modern Philosophy. He wrote his famous quote “I think, therefore I Am.” while using his “Method of Doubt” to decide what he knew for sure by doubting everything until he arrived at something that was indisputable. We use a similar approach today by suspending belief in mathematical ideas until we can prove them.)

At this time, have students work in groups on **Activity 6.1.7 Special Angles and Arc Lengths** thatnot only introduces students to the radian measure for the angles in the special triangles they learned in geometry – the 30-60-90 degree and the isosceles right$ $triangles - but also provides students with practice and drill with fractions. Students will learn the radian measure and the degree measure for these special angles by dividing unit circles, 2π and 360° into quarters, sixths, eighths and twelfths. Allow students to find their own organizational structure for writing the radian measures and degree measures for each angle. Patterns will emerge. Example: given that 2π radians = 360°, one twelfth of a circle is π/6 radians = 30°. For radian measure, use exact values in both the simplified and non-simplified fractions: e.g. use 1π/6, 2π/6, 3π/6, 4π/6 …12π/6 as well as simplified fractions. If your students enjoy using geometry software, they may do this activity with Geogebra or other geometry software.

**Group Activity – Activity 6.1.7** introduces students to the radian measure for the angles in the special triangles they learned in geometry – the 30-60-90 degree and the isosceles right$ $triangles - but also provides students with practice and drill with fractions. Students will learn the radian measure and the degree measure for these special angles by dividing unit circles, 2π and 360° into quarters, sixths, eighths and twelfths.

Introduce the word ‘Quadrantal angles’ that are the angles that result from partitioning the circle into quarters. The terminal ray of a quadrantal angle lies on the x or y axis.

This activity works well using the group learning technique ‘Jigsaw Puzzle’. From each student group, pick one student to work on halves and quarters, and re-group these students so they can work together. Then pick another student from each group to do the eighths, and have these people work together; similarly for the sixths and twelfths. You can group the students that work slowly into halves and fourths, and the students that work quickly and understand fractions best into the twelfths splinter group. Consider having each member of the group draw their circles on clear plastic so that when the original groups come back together, the clear sheets can be layered on top of each other so students can see how the fractional parts of the circle coincide. After working in splinter groups, the students return to their original group where each student is responsible for teaching their partitioning to the others.

To complete this activity, tell students to make a final clean copy of a unit circle that is partitioned into quarters, sixths, eighths and twelfths and color coded so that each partitioning of the circle has the same color. Sixths will be doubly colored. Indicate the arc length and central angle measure at the intersection of the terminal ray and the circle. The measures should be exact radians in terms of π, decimal approximation, and degrees. Be sure students save this circle, because they will label the coordinates of each point of intersection when they study the Wrapping function in investigation 2. The circle will become their reference tool for the trigonometric values of the special angles.

**Differentiated Instruction.** Allow students to develop their own method for learning the position of angles and arc lengths such as 5π/6 on the unit circle. Some students will want to think dividing the unit circle in fractional parts such as twelfths in this example, and then counting 5 of these parts. Other students may see 5π/6 as one π/6 short of the half circle π. Others may divide the upper semicircle into sixths. Some students will have memorized the positions of these angles on the circle. Once students learn to convert radians to degrees using the conversion factor 180°/π, they may prefer to “translate” every angle into degrees and work in degrees. Encourage the students to develop different strategies.

**Activity 6.1.8 Convert Between Degrees and Radians** has students learn the conversion factor (180°/π radians) for radians to degrees and also from degrees to radians (π radians/180°). Students will perform a variety of conversions and sketch the angles in standard position. Angles larger than 2π are included as well as angles in radian measure that do not include the symbol π (e.g. “convert 3 radians to degree measure”). The activity includes a review of graphing angles in standard position, then moves on to having student graph angles that are larger than 2π . Students will see how the coordinates repeat themselves every 2π, and how the odd/even symmetry of the circle helps them find corresponding points.

**Activity 6.1.8a** is identical to **6.1.8** except that the first page of Activity **6.1.8a** has more extensive instruction and practice for converting various units of measure.

Before distributing **Activity 6.1.8 Convert Between Degrees and Radians,** use a 10 minute class discussion to work through one example of graphing an angle, say, 1560°. Convert to radian measure. The discussion will give you the opportunity to develop students’ vocabulary by using (and defining if necessary) words such as initial ray, terminal ray, co-terminal point, period and to explain what is meant by an angle or arc in standard position. Label the graph of the angle with the vocabulary words.

**Differentiated Instruction.** Before students begin Activity 6.1.8, review conversion factors and unit analysis with students. Show, for example, that since 12 inches = 1 foot, then 12 inches/1 foot = 1 (why?). If we want to convert 4.5 feet to inches we multiply 4.5 feet (12 inches/1foot) the unit of measure “feet” cancels, the unit “inches” is what is left. The numbers 4.5 and 12 are multiplied. Use **Activity 6.1.8a** if students need more practice with converting units of measure using this factor method. Otherwise, use **6.1.8**.

**Differentiated Instruction.** Some students may prefer to convert between degrees and radians by using proportions: To convert 5π/6 to degrees, think: 360° is to 2π radians as x° is to 5π/6 radians. Solve: $\frac{360°}{2π}= \frac{x°}{\frac{5π}{6}}$ . Allow students to write the proportions in any of the various correct forms: $\frac{\frac{5π}{6} }{2π}= \frac{x°}{360°}$ for example.

**Journal Prompt 2** asks students to explain how to convert radians to degrees and degrees to radians. Explain how to decide which factor to use. Use the conversion factor to find approximately how many degrees is one radian. Sketch an arc and a central angle that measures 1 radian. Students might respond with the proportion method first. Then seeing the second request add the dimensional analysis method.

**Closure Notes**

At the end of Unit 6, Investigation 1, students should be comfortable with the geometry and the algebra of the unit circle, and with the measures of central angles in radians and degrees. Tell students that we are now going to use the circle, angles, and the points where the arc ends in order to generalize the right triangle trigonometry to apply to all real numbers and to allow for modeling certain periodic behaviors. To put the students in the correct mindset for Investigation 2, challenge them to explain how the features of circles and imbedded right triangles are connected –i.e. when the acute angle of the triangle is at the origin. If they need to be steered in the right direction, you can put a circle on the board and draw a central angle, asking them if they can see where a right triangle would fit; and you can draw a right triangle and then place a circle around the right triangle so one of the acute angles is at the origin and one of the legs is on the x axis. Students could make the following observations: the radius of the circle is the same as the hypotenuse of the circle; a height of the triangle (opposite leg) is the y coordinate of the point where the terminal ray of the angle intersects the circle; the triangle “grows and shrinks and grows and shrinks” as the angle rotates around the circle.

**Vocabulary**

Angle

Angle measure

Arc of a circle

Arc length

Central angle

Circumference

Constant of proportionality

Co-terminal angles

Conversion factor

Degrees
Initial ray

Periodicity

Quadrantal angle

Radian

Similarity

Special angles and special triangles

Standard Position (as in: angle in standard position on the coordinate plane)

Subtend (as in: arc subtended by angle)

Terminal ray

Unit analysis

Unit Circle

**Resources and Materials**

**Activities 6.1.1, 6.1.2, 6.1.5, 6.1.6, 6.1.7 and 6.1.8 or 6.1.8a should be completed in this investigation**

Activity 6.1.1 Angles Measure Can Be Any Number of Degrees

Activity 6.1.2 What is a Radian?

Activity 6.1.3 Learn More About Radians and Arc Lengths

Activity 6.1.4 Arc Length as Direct Variation

Activity 6.1.5 Radian Measure of Central Angles as the Ratio $\frac{s}{r}$

Activity 6.1.6 The Unit Circle

Activity 6.1.7 Special Angles and Arc Lengths

Activity 6.1.8 Convert Between Degrees and Radians

Activity 6.1.8a Convert Between Degrees and Radians

Graphing calculator computer software with a graphing utility

Ruler with centimeter markings and inch markings

Compass

Protractor

Tape measures with centimeter and inch markings (or have students make their own using string to mark of the arc length, and then lay the string on the ruler to)

String, twine (not ‘jute’), thin ribbon, pipe cleaners or waxed twine (like the craft toy “bendables)

Markers for marking the string or ribbon

Scissors

4 different colored pencils, markers or pens

Tracing paper or Clear plastic sheets to write on, and water soluble markers – such as those used with an overhead projector

Geometry Software such as Geogebra

Sine Tracer applets or directions for creating a Sine Tracer on a Calculator.

Embeddedmath.com for fill ins for the unit circle and filled in answers as well. Supports Activity 6.1.7.