**Unit 6: Trigonometric Functions**

**UNIT OVERVIEW**

19 days

In unit 6, students will study periodic behavior and periodic functions, specifically the circular definition of the sine, cosine and tangent functions. This unit builds on the right triangle trigonometry learned in geometry by using the unit circle to extend the domain of the trigonometric functions from acute angle measures in geometry to all real numbers. Investigations familiarize students with degree and radian measure of angles of rotation before they study the wrapping function followed by the definitions of the sine, cosine and tangent in terms of the coordinates of the terminal point of an arc on the unit circle. Students will understand how the periodic behavior of the trigonometric functions is determined by their circular definition.

Students discover that the circular definitions are extensions of right triangle trigonometry by sketching the reference triangles for various angle measures. They observe, for example, that the measure of the vertical leg of the triangle is the y coordinate of the terminal point of the arc, and that the sine of the reference angle is the ratio of the lengths of the leg opposite the central (reference) angle to the length of the radius/hypotenuse that is 1. Particular attention is paid to the special right triangles 45°-45°-90° and 30°-60°-90° measured in both degrees and radians.

By plotting points using raw spaghetti as a manipulative and interacting with technology based sine-tracer applets students will visualize the graph of the sine function as the vertical height of the terminal point of an arc as the arc wraps around a circle. Similarly, the cosine is the horizontal distance of the terminal point from the y axis, and tangent is the ratio of the y coordinate to the x coordinate that is also the slope of the terminal ray of the central angle subtending the arc of length t. Students will understand that features of the circle influence features of the trigonometric functions’ graphs. For example, that the period of the sine and cosine functions is 2π, and the period of the tangent function is π; that the radius of the circle is the amplitude of the graph of the sine and cosine functions; and that sin(t ± π) = -sin(t), and cos(t ± π) = -cos(t).

Examples for modeling with trigonometric functions occurs throughout Unit 6, so students will have experience modeling with trigonometric functions. By the time they come to Investigation 5, Models of Periodic Behavior, students may be able to work through three or four of the modeling activities with little direct instruction from the teacher. Instead of direct instruction, the teacher is encouraged to ask pointed or probing questions, and encourage students to figure things out as much as possible on their own and by working with other students. Providing this opportunity for students to apply what they learned in novel contexts will develop their abilities in the eight Mathematical Process Standards of the Common Core.

The unit concludes with the basic Pythagorean Identity (sin *x*)2 + (cos *x*)2 = 1, and, if time, the angle sum identities. In addition to learning this important identity, students will have an opportunity to practice symbolic manipulation.

**Investigation 1** introduces students to angles as revolutions and measuring a central angle in degrees and radians.

By wrapping string around a unit circle, students will discover:

* the definition of radian,
* the equivalence of arc length on a unit circle and the radian measure of the central angle that subtends the arc: s = 𝜽 for r = 1 unit.

Students will focus especially on arc lengths and central angles for integer multiples of π, π /2, π /3, π /4, and π /6 and they will learn how to convert between degrees and radians. Finally, they will use the algebraic representation of the unit circle to find points on the circle and graph the circle on the coordinate plane.

Note: this investigation does not use the wrapping function per se, since we are not finding the coordinates of the end point of the arcs until Investigation 2.

**Investigation 2** has students develop the unit circle definition of the trigonometric functions, starting with the wrapping function W(t) that maps an arc length t to the coordinates of the terminal point of the arc in standard position. W(t) indicates the point on the unit circle that is determined by both a) the point of intersection of the unit circle and terminal ray of angle ‘t’ in standard position and b) the end point of an arc of length ‘t’ that has (1,0) as the initial endpoint. Students learn that the sine, cosine and tangent of t are the y, x and y/x of the point W(t). By sketching the angle, arc, and corresponding right triangle, students will see the relationship between the right triangle definition and the circular definitions of the trigonometric functions.

Using the odd and even symmetry of the unit circle, and given the value of the trigonometric function of t, students will be able to find the trigonometric function of t added to an integer multiple of π.

**Investigation 3** Now that students know the circular definition of the trigonometric functions with a domain of all real numbers, they graph the basic trigonometric functions. Students will use lengths of uncooked spaghetti to transfer the vertical distance on the unit circle associated with given angular measure to a coordinate plane to form the graph of the sine function. Similarly, using spaghetti strands to mark the horizontal distance of a point on the circle and plotting the x coordinate of the point, students could create a graph for y = cos(x). Students will see that the tangent value is the slope of the terminal ray of angle and that each vertical asymptote on the graph of y = tan(t) corresponds to an angle whose terminal ray has a vertical slope. A “Sine Tracer” animation or applet will help students see how the y component of the coordinate W(t) determines the graph of y = sin(t). Observing the height of a person riding a variety of Ferris Wheels as a function of time provides students with a real world interpretation for the sine function and foreshadows the translations and transformations of the graph of the sine function.

**Investigation 4** Students will apply what they have learned about the transformations of functions in previous units to trigonometric functions. They will learn about the amplitude, period and midline of trigonometric functions as they are associated with the vertical and horizontal stretches, and vertical shifts. An activity for STEM intending students will help them learn about the phase shift for trigonometric functions.

**Investigation 5.** Students will model periodic behavior for functions where the independent variable is time and the dependent variable is the number of hours of daylight per day, heights of ocean waves, temperatures, distance of a horizontal distance of a pendulum from a fixed center point, to name a few examples. In the case of sound or musical notes, graphing air pressure as a function of time will give a sinusoidal curve.

**Investigation 6.** By applying the Pythagorean Theorem to a right triangle with angle measure ‘t’ in standard position and re-naming the sides of the triangle, students will deduce the Pythagorean Identity: (sin t)2+ (cos t)2 =1. Students will be able to use the identity to find two of the trigonometric values given the third trigonometric value and the quadrant of angle t. STEM – intending students will also prove that sin(a + b) ≠ sin(a) + sin(b), and will prove the angle sum identities.

**Essential Questions**

* What are the advantages of using radians to measure central angles and the corresponding subtended arc?
* How do we use circles to extend the domain of the trigonometric functions learned in geometry from acute angle measures to trigonometric functions of any real number?
* For angles of measure t in standard position whose terminal ray intersects the unit circle at point W(t), why is the cosine of the angle defined as the x coordinate of W(t), the sine of the angle as the y coordinate of W(t) and the tangent of the angle as the slope of the terminal ray of the angle?
* Why are the periods of the sine and cosine functions 2 and the period of tangent π?
* Why is the range of the sine and cosine functions -1 ?
* Why does the tangent function have vertical asymptotes?
* How do we use sums of sinusoidal functions and translations and transpositions of trigonometric functions to model periodic behavior?
* What algebra properties and techniques that we have learned so far apply to simplifying trigonometric expressions and solving trigonometric equations?

**Enduring Understandings**

* With real data, if the data repeats in a pattern, a trigonometric function may provide an appropriate model.
* The trigonometric (circular) functions can be defined over the real numbers.

**Unit Understandings**

* The sine, cosine, and tangent functions can have a domain of (-∞ᵒ, ∞ᵒ) or of (-∞, ∞).
* The arc length and the radian measure of the central angle that the arc subtends on a unit circle are equal: “s = t.”
* The sine and cosine of t can be interpreted as the vertical and horizontal directed distances, of W(t) from the x and y axes, respectively.
* The tangent of t can be interpreted as the slope of the terminal ray of angle t in standard position on a unit circle.
* The usual algebraic manipulations apply to expressions involving the trigonometric functions such as sin(x) + sin(x) = 2 sin(x).  However algebraic manipulation of trigonometric functions may permit further simplification using trigonometric identities.
* A sinusoidal model can be obtained for data if the period, maximum, and minimum function values can be determined.
* One can find the other representations of a trigonometric function given one of the following representations: algebraic formula, graph, tables, verbal description.

**Unit Contents**

Investigation 1: The Unit Circle and Radian Measure (4 days)

Investigation 2: Unit Circle Definition of Trigonometric Functions (3 days)

Investigation 3: Graphs of Trigonometric Functions (2 days)

Investigation 4: Transformations of Trigonometric Functions (2 days)

Investigation 5: Models of Periodic Behavior (2 days)

Investigation 6: Trigonometric Equations and Identities (3 days)

Performance Task: Find a Trig Function (1 day for presentations)

Review for Unit Test (1 day)

End of Unit Test (1 day)

**Common Core Standards**

*Mathematical Practices #1 and #3* *describe a classroom environment that encourages thinking mathematically and are critical for quality teaching and learning. Practices in bold are to be emphasized in the unit.*

1. **Make sense of problems and persevere in solving them.**

2. Reason abstractly and quantitatively.

3. Construct viable arguments and critique the reasoning of others.

4**. Model with mathematics**.

5. Use appropriate tools strategically.

6. Attend to precision.

7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning.

**Common Core State Standards**

F.TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F.TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for π/3, π/4 and π/6, and use the unit circle to express the values of sine, cosine, and tangent for π - x, for π + x and 2π - x in terms of their values for x, where x is any real number.

F.TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.\*

F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F.TF.8 Prove the Pythagorean identity (sin A)2 + (cos A)2 = 1 and use it to calculate trigonometric ratios.

F.TF.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

F.BF.3 Identify the effect on the graph of replacing *f*(*x*) by *f*(*x*) + *k*, *kf*(*x*), *f*(*kx*), and *f*(*x* + *k*) for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales

**Assessment Strategies:**

**Performance Task**

The Performance Task “Find a Trig Function” asks students to find data for a real world situation that can be modeled, even if only approximately, by a trigonometric function. Using the mathematical model for their data, students will analyze the real world phenomenon. Some ideas for students to investigate are a) the fluctuation of predator and prey populations over time, b) the phases of the moon and the height of the high tide over the course of a year, c) biorhythms or circadian rhythms, d) extreme tides such as those in the Bay of Fundy, or d) how a rotating light on a police car creates a spot of light that moves along a wall (students could re-enact this behavior by shining a flashlight on a wall and rotating it. For each eighth of a rotation, determine the distance between the spot of light on the wall and a given starting point.)

For those that wish, they can do the performance task called “Musical Notes” in order to develop a theory of consonance and dissonance: what characterizes notes that sound good together and those that do not? Though the answer to this question is a matter of personal taste, many guidelines for “what sounds good” have been espoused and modified since man first sang a musical interval. Student will learn about the Pythagorean intervals by placing a finger on the string of a guitar (or other stringed instrument) at various fractional distances from the bridge, plucking the string and finding what lengths of string produce notes that are an octave, a fifth, a fourth and a third above the open string. They will write the sine function for some of the notes on the musical scale where the independent variable is time, and the dependent variable is pressure fluctuation generated by plucking the string, e.g. Students will investigate consonant and dissonant intervals by exploring the sum of the sine functions that model various intervals such as the octave and the fifth.

A lesson is provided for students who may want to compare the Pythagorean scale based on fractions with the equal-tempered scale learned in Unit 1 where the frequencies of the notes form a geometric sequence with a common ratio equal to the twelfth root of 2.

**Other Evidence (Formative and Summative Assessments)**

* Exit slips
* Class work
* Homework assignments
* Math journals
* Unit 6 end of unit assessment

**Vocabulary**

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| Amplitude  Angle  Angle measure  Arc of a circle in standard position  Arc length  Central angle  Circular definition of the trigonometric functions  Circumference  Constant of proportionality  Constructive and destructive  Conversion factor  Co-terminal angles  Decimeter  Degrees  Initial ray  Inside and outside changes  Interference  Longitudinal wave  Mathematical model  Maximum  Midline  Minimum  Odd and even symmetry  Parameter | Periodic behavior  Period of a function  Phase Shift (+)  Quadrantile angles  Radian  Reference angle  Reference triangle  Right triangle definition of trigonometric functions  Similarity  Sinusoidal  Special angles  Special triangles  Standard Position (as in: angle or arc in standard position on a unit circle in the coordinate plane)  Subtend (as in: arc on circle subtended by angle  Terminal ray  Transformations of functions  Transverse wave  Trigonometric functions of real numbers  Unit analysis  Unit Circle  Wrapping Function |