**Unit 4: Investigation 3 (4 Days)**

**Graphs of Rational Functions**

**Common Core State Standards**

A.SSE.1b Interpret complicated expressions by viewing one or more parts as a single entity.

A.APR.1bRewrite simple rational expressions in different forms; write a(x)/b(x) in the form q(x) + r(x)/b(x), where a(x), b(x), q(x), and r(x) are polynomials with the degree of r(x) less than the degree of b(x), using inspection, long division, or, for more complicated examples, a computer algebra system.

A.APR.7(+)Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities and sketch graphs showing key features given a verbal description of the relationship.

F.IF.5Relate the domain of a function to its graph and where applicable, to the quantitative relationship it describes.

F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

F.IF.7d(+)Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, by table, or verbally)

F.BF.3Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k ( both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

A.REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve.

**Overview**

This investigation focuses on graphing rational functions, given a defining equation by noting the domain, vertical asymptotes**,** horizontal asymptotes, x- and y- intercepts, end behavior, behavior near a vertical asymptote and holes. Students will formulate, after examining the graphical behavior of rational functions defined by more complicated expressions, a rule for determining if a rational function will have a horizontal asymptote or not. For STEM intending students, additional activities are provided that include functions with holes or that have an oblique asymptote.

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Explain why the graphs of some rational functions have horizontal asymptotes.
* Determine the equation of a horizontal asymptote when it exists.
* Explain why the graphs of some rational functions have vertical asymptotes.
* Explain why the graphs of some rational functions have a hole. (+)
* Determine the equation of a vertical asymptote.
* Interpret the meaning of a vertical or horizontal asymptote in a real world setting.
* Determine the location of a hole in a graph. (+)
* Make quick sketches of rational functions of the form f(x) = (ax + b) /(cx + d).

**Assessment Strategies: How Will They Show What They Know?**

* **Exit slip 4.3.1** asks students to graph a rational function and to identify any horizontal or vertical asymptotes, the domain, x- and y-intercepts, zeros of the function.
* **Exit slip 4.3.2** asks students to define a function that has a specified vertical and/or horizontal asymptote.
* **Exit slip 4.3.3(** + **)** asks studentsto define a function f so that its graph has a hole instead of a vertical asymptote at the excluded value.
* **Journal Prompt 1** asks studentsto explain what conditions must be met to have y = 0 be the equation of a horizontal asymptote and to determine an equation for a rational function that has a horizontal asymptote with equation y = 3/2 .
* **Journal Prompt 2** asks students to summarize the graphical behaviors of rational functions, noting how to obtain x – and y – intercepts, determine end behavior, equations of asymptotes and any other characteristics they wish to include and to illustrate their observations with equations or graphs if they care to.
* **Activity 4.3.1 Graphing Rational Functions I** will have them examine translations of rational functions of the form f(x) = a/(x – h) + k and relate the roles of a, h and k to their roles in g(x) = a(x – h)2 + k and h(x) = a│x - h│+ k. A first definition for vertical and horizontal asymptotes will be developed after some exploration of the behavior of functions near these lines. Students will develop the connection between excluded values and vertical asymptotes, and the connection between end behavior and horizontal asymptotes.
* **Activity 4.3.2** **Graphing Rational Functions II** will have them observe that rational functions can have more than one equation form and they will examine the quotient form for a rational function. They will also find x- and y-intercepts and start some modest manipulation of rational expressions.
* **Activity 4.3.3** **Graphing Rational Functions III** contains rational functions whose graphs do not have a horizontal asymptote as well as functions whose graphs cross their horizontal asymptote. It also has functions whose graphs have two vertical asymptotes and functions whose graphs have none. Holes (+) in a graph are also explored on pages 5 and 6.
* **Activity 4.3.4** **Applications of Rational Functions** contains expressions with like denominators for combining and some division by monomials. There are also some word problems that need a rational function model.
* **Activity 4.3.5 Graphing Rational Functions IV** will have students compare rational function behavior with quadratic, exponential and cubic function behaviors, hopefully seeing that f(x) =*a*g(x) vertically stretches, f(x) = a + g(x) translates vertically and f(x) = g(a + x) translates horizontally regardless of the underlying function family. Zeros, asymptotes and x-and y-intercepts will be noted.
* **Activity 4.3.6 (+)Graphing Rational Functions V** continues to study graphs with holes and makes more formal the conditions under which there is a hole in the graph rather than a vertical asymptote. The activity also includes functions whose graphs have oblique asymptotes.
* **Activity 4**.**3.7 (+) Queueing Theory Application** will have students examine a steady-state model of a single server queueing system and interpret the end behavior, the roles of the asymptotes, the domain in terms of the application.

**Launch Notes**

Before using **Activity Sheet 4.3.1** ask students to consider f(x) = a(x – h)2 + kand the roles of a, h and k and again to considerh(*x*) = a│*x* - h│+ k. Then ask them if they remember a horizontal asymptote from Algebra 1when graphing an exponential function. What was special about this horizontal line? Stress that a horizontal asymptote characterizes end behavior. Stay informal. Also ask them about the graph of f(*x*) = 1/*x* from the last investigation and tell them they will examine the end behavior and local behavior around x = 0 more carefully in the next activity. Using **Activity Sheet 4.3.1** **Graphing Rational Functions I** have students individually, in pairs, or in groups work through the examples.

**Teaching Strategies**

By the end of the lesson students should have a class definition of a vertical asymptote, a horizontal asymptote, and they should have observed and be able to articulate a relationship between an excluded value and the existence of a vertical asymptote and the end behavior and a horizontal asymptote. Some of the examples in this activity can be reserved for homework . When **Activity 4.3.1** is completed, students should summarize how the values excluded from the domain of the function can assist with locating vertical asymptotes and how the equation of a horizontal asymptote is related to the expression a/(x - h) + k. Pose the question, “As x gets larger and larger the denominator gets bigger and bigger and the rational expression approaches what value ?” A similar argument can be made for negative x values that get smaller and smaller.

Once students demonstrate an understanding of the roles of the parameters in the defining equation and their relationship to a vertical asymptote and a horizontal asymptote, they can move to **Activity 4.3.2 Graphing Rational Functions II** which will have them observe that rational functions can have more than one form and they will examine the quotient form for a rational function. They will also find x- and y-intercepts and start some modest manipulation of rational expressions. If using a grapher, have students put f(x) = 1 + 1/xin Y1= and g(x)= (x + 1)/x in Y2 = and graph the functions. Students should explain why they see only one graph. Do several more. Then have them explain how using division you can change (5x + 1)/x into 5 + 1/x and how by using a common denominator get (5x + 1)/x from 5 + 1/x. Do the same with some other examples. Have students combine a few more expressions and divide a few more by the monomial divisor. Then have students consider f(x) =( x – 3 )/(x – 5). Discuss the domain, x – and y - intercepts, and asymptotes of the graph. Break into groups after the discussion and give each group a rational function **from Activity 4.3.2** to analyze and graph. Each group should present their analysis and graph to the class. Examples include rational functions of the form f(x) = (ax + b) / (cx + d).

**Group Activity 4.3.2 Graphing Rational Functions II** Each group can analyze their function: domain, vertical asymptotes, end behavior, horizontal asymptotes, x – and y – intercepts and graph, and then present to whole class. As a group presents, members of the other groups can fill in the information on their activity sheets.

When **Activity 4.3.2** group work is completed, students should summarize how the values excluded from the domain of the function can assist with locating vertical asymptotes on the graph and how the equation of a horizontal asymptote is related to the expression (ax + b) /(cx + d) and the end behavior of the function. Pose the question, “As x gets larger and larger the denominator gets bigger and bigger and the expression approaches what value ?” A similar argument can be made for negative x values that get smaller and smaller. Then have them examine the expression (ax + b)/(cx +d). They need to appreciate that as x gets bigger and bigger the contributions of b and d are negligible. Thus it is the behavior of the expression ax/cx that is critical to determining the equation of the horizontal asymptote. Ask students if they can find a shortcut to assessing the end behavior. Ask students to explain what conditions must be present to have a horizontal asymptote or a vertical asymptote. **Exit slip 4.3.1 can be used here**.

**Differentiated Instruction (For Learners Needing More Help)** To analyze graph f(x) = 1/(x2 – 4) in Activity 4.3.2 you might have students start with a factored denominator first, then multiply the factors (x - 2) (x + 2) so students can see that the denominator could have been written as either expression. Making a table of values with their graphers can also give the students an appreciation for the extreme changes. Using the table in auto mode and starting at - 3 and going up by 0.1 may provide a very concrete representation of the behavior of this function.

Ask mode can be used once the students get to -2.1 and students can enter -2.01,-2.001, then -1.999, -1.99. Or GeoGebra can be used to create a table of values in which the x-values go up by a power of 1/10.

**Activity sheet 4.3.3** asks students to analyze the behavior of graphs of functions that cross the horizontal asymptote and then gives them to opportunity to think harder about what it means to be a horizontal asymptote. It also has rational functions that do not have horizontal asymptotes and ones that do not have a vertical asymptote. The question can then be raised: Can we determine when and where a function’s graph will cross its horizontal asymptote? The need to solve rational equations should emerge and will be addressed in investigation 5.

Holes are a STEM topic but it is recommended that you consider the graphs of the functions in the next paragraph, and #11 on the activity sheet with all students so at least they are aware that holes can exist.

Have students consider the graph of f(x) = (2x – 2 )/(1 – x) . Before having them graph the function, ask them what they expect and to even make a very quick rough sketch of the type of graph they expect. Then have them graph the function. Are they surprised? One is an excluded value. Pose the question: Why did they not get an asymptote? Have them try f(x) = (2x – 4 )/(x2 – 4) and then f(x) = (x2 + 4x)/x. Try to have them determine the condition one must meet to have a hole instead of a vertical asymptote.

For non-STEM intending students it is not necessary to complete # 12- 14 on pages 6 of **Activity Sheet 4.3.3.** The end of unit assessment will not have holes. For STEM-intending students question 13 on the end-of-unit assessment can be replaced with a function that has a hole in its graph.

**Journal Prompt 1** From your experience so far, what conditions must be met to have y = 0 be the equation of a horizontal asymptote? Determine an equation for a rational function that has a horizontal asymptote with the equation y = 3/2.

Before proceeding to more complicated rational expressions that can be used to define a rational function, have students try some applications problems. For example, pose the following problem: Ann got an 82 on her first quiz. She wants to have an average of 95. A) If she can get a 97 on each succeeding quiz, can she get a 95 for the semester? How many quizzes would the teacher have to give after that first quiz? B) Suppose she thinks she can get a 100 on each succeeding quiz. How many quizzes would the teacher need to give now so she can have a 95 average ? **Activity 4.3.4** **Applications of Rational Functions,** can be started in class and completed for homework. **Exit slip 4.3.2** can be distributed. For Stem-intending students you might prefer **Exit Slip 4.3.2+.**

**Activity 4.3.5 Graphing Rational Functions IV** is intended to have students again examine some of the behaviors possible when graphing a rational function and depending upon the form of the defining expression use transformations as well. It also has students graph earlier studied functions using transformations. For example the functions f(x) = 1/(x2 + 4), g(x) = x/(x2 + 4), h(x) = 2x2/(x2 + 4) each have the domain of all real numbers but the equations of the horizontal asymptotes can differ and the symmetry can be odd or even. Students will be asked to complete their generalization on how to predict the end behavior of the function: will there be a horizontal asymptote and if so will its equation be y = 0 or y = a/b.

For Stem intending students **Activity 4.3.6 Graphing Rational Functions** V(+) has more practice with graphs with holes. The activity is intended to have students compare and contrast and also by the end draw conclusions about a hole vs. vertical asymptote( +) . A few functions will have oblique asymptotes (+) and a few will cross their horizontal asymptotes. Students need only be aware of oblique asymptotes, not experts in finding them.

**Differentiated Instruction (Enrichment)** Students can examine additional functions that have or do not have oblique asymptotes as they go off to infinity. Then they need to determine by just examining the equation of a rational function, those that have an oblique asymptote and those that have none. They need to note that if f(x) = g(x)/h(x), g(x) must have a degree one greater than the degree of h(x) and that after dividing g by h, f(x) after division will equal (mx+ b) + k(x)/h(x) where the degree of k will be less than the degree of h.

**Journal Prompt 2** Summarize the graphical behaviors of rational functions, noting how to obtain x – and y – intercepts, determine end behavior, equations of asymptotes and any other characteristics you would like to comment on. Feel free to illustrate with equations or graphs. Answers will vary but ideally students will include that we find the y-intercept by substitution (let x be 0) as we did for the other function families and that to find an x intercept we need to solve k(x) = 0 and for a rational function whose defining form is f(x)/g(x) that means we set f(x) = 0 and solve. They will note that a rational function may have a domain of all reals or have excluded values and that often at the excluded values there will be asymptotic behavior and a vertical asymptote should be included in the graph of the function to denote this behavior. They will also note the end behavior which can be described by a horizontal asymptote or it may include going to ∞ or -∞. They might note that there can be symmetry and provide examples or that the graph of a rational function can cross its horizontal asymptote but never its vertical asymptote.

**Closure Notes**

Use the journal 2 responses to summarize the graphical behavior of rational functions and when discussing crossing a horizontal asymptote, use this as an opportunity to promote the need to be add, subtract, multiply and divide rational expressions and to use these skills to solve rational equations. Where appropriate, draw attention to how rational expressions form a system like rational numbers. Tis will set the stage for the last two investigations of this unit.

**Vocabulary**

Domain

Excluded value

Range

Vertical asymptote

End behavior

Horizontal asymptote

Vertical translation

Horizontal translation

Oblique asymptote

Hole

x – intercepts

y - intercepts

**Resources and Materials**

**Activities 4.3.1, 4.3.2, 4.3.3, 4.3.4, 4.3.5 should be completed in this investigation. Pages 5 and 6 can be omitted in activity 4.3.3 for non-stem intending students but it is recommended that all students if time permits complete the entire activity.**

Activity 4.3.1Graphing Rational Functions I

Activity 4.3.2Graphing Rational Functions II

Activity 4.3.3Graphing Rational Functions III

Activity 4.3.4 Applications of Rational Functions

Activity 4.3.5Graphing Rational Functions IV

Activity 4.3.6(+)Graphing Rational Functions V

Activity 4.3.7 (+) Queueing Theory Application

Graphers or equivalent software

Graph paper

Mathematics Teacher “Queueing Theory: A Rational Approach to the Problem of Waiting in Line V5, No. 5 , May 2012