**Unit 1: Investigation 3 (4 Days)**

**Types of Functions**

**Common Core State Standards**

A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the* *function h(n) gives the number of person-hours it* *takes to assemble n engines in a factory, then the* *positive integers would be an appropriate* *domain for the function*.\*

F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

F.IF.7b Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.

**Overview**

At this point in their mathematical careers, students have only studied linear and possibly quadratic or exponential functions in a careful and systematic way. This investigation is the starting point for students to consider other types of functions, including quadratic, exponential, absolute value, step, piecewise, and power functions (y = *xn*). The starting point for this is a consideration of the growth of functions, since growth is one of the primary ways to distinguish families of functions. Considering function growth also sets the stage for future mathematical concepts in precalculus and calculus. Students will also see that not all functions are “smooth” and “nice”; it is absolutely possible for a function to have “more than one piece.” Indeed, piecewise functions and step functions have very important applications. Students will explore the symmetry of functions and the concept of odd and even functions, making connections back to the ideas of symmetry that form an important part of Geometry.

**Assessment Activities**

**Evidence of success: What will students be able to do?**

* Be able to determine if a function is linear, quadratic, exponential, or some other type of function by investigating its growth.
* Be able to graph absolute value functions and to understand their properties.
* Provide examples of functions that are not “smooth” and “in one piece,” such as piecewise functions and step functions.
* Be able to graph piecewise and step functions, and to describe their properties.
* Determine if a function has odd or even symmetry.

**Assessment tools: How will they show what they know?**

* **Exit Slip 1.3** asks students to determine the type of growth a function exhibits from a table of values.
* **Journal Prompt 1** asks students to describe the properties of different types of functions, including linear, quadratic, exponential, absolute value, and piecewise functions, and to explain if any of these exhibit odd or even symmetry.
* **Activity 1.3.1 Linear and Nonlinear Growth** asks students to take a number of functions and classify them as to the type of growth they exhibit. At this point in the curriculum, if a function is not linear, exponential, or quadratic, it is considered to be “none of the above.”
* **Activity 1.3.2 The Absolute Value Function** introduces students to the properties of this function. They graph absolute value functions and use absolute value to find the ideal location for the ideal location of a warehouse based on the total distance (absolute value) from a number of cities.
* **Activity 1.3.3 Piecewise and Step Functions** presents students with a range of realistic problems that give rise to piecewise and step functions. Students create both graphs and tables for such functions.
* **Activity 1.3.4 Symmetry Review** provides a number of figures, including some corporate logos. Students determine whether the figures have line and/or rotational symmetry.
* **Activity 1.3.5 Even and Odd Functions** asks students to consider a range of power and polynomial functions and consider their symmetry with respect to the y-axis and the origin; they will see that polynomials with all even exponents have symmetry with respect to the y-axis (even symmetry), while polynomials with all odd exponents will have symmetry with respect to the origin (odd symmetry).

**Launch Notes**

Start this Investigation by showing the following two-minute video (it is possible to bypass the first minute if time is an issue): <https://www.youtube.com/watch?v=DjlEJNfsOKc>. In the video, there is an array of ping pong balls on top of a set of mouse traps. A student drops one new ping pong ball on top of one of the others, which causes that ball to launch, which causes several more to launch, etc., until most of the balls have been launched from the traps. As the speaker points out when this action is shown in slow motion, if you make a graph of the number of ping pong balls in motion as a function of time, the graph is not linear—the number increases more and more quickly.

Point out that Algebra 1 focuses primarily on linear functions. While these are very important functions in mathematics and elsewhere, they are only one type of function. In this investigation, we will look at other types of functions, especially considering the way that they grow as the independent variable increases.

**Teaching Strategies**

In Investigation 3, students will examine how functions grow and will “sort” functions into four groups according to their growth—linear, exponential, quadratic, and other. You may wish to take the first problem in **Activity 1.3.1: Linear and nonlinear growth** and work through it as a whole class. As you do so, fill in the first blank column and ask students what kind of a function would have a constant difference in outputs. Students should remember from Algebra 1 that linear functions exhibit this kind of growth. Now fill in the second column (successive ratios between outputs) and point out that functions where this column is constant exhibit exponential growth. If they studied exponential functions in Algebra 1 they would have seen this. Next, fill in the third column (successive second differences in outputs) and point out that functions where this column is constant exhibit quadratic growth. Finally, fill in the fourth column (successive ratios between the difference in outputs) and point out that functions where this column is constant also exhibit exponential growth. There are names for other patterns in such tables, but for now we will simply call them “other” types of growth. Once you have worked through a single table together as a whole group, you can have students complete the others individually.

It is important to distinguish at this point between types of function growth and types of functions. For example, while we can conclude that the table in question #2 of Activity 1.3.1 exhibits linear growth, we cannot correctly conclude that the data comes from a linear function, because there are other functions that could fit the data in this table. For example, the polynomial function g(x) = x6 – 21x5 + 175x4 – 735x3 + 1624x2 – 1766x + 725 fits the table of values in question #2 as well.

**Differentiated Instruction (Enrichment)**

The question of finding a polynomial that fits a table of values, called polynomial interpolation, is an important topic in more advanced mathematics courses such as numerical analysis. An interested student (or teacher) could find out more about polynomial interpolation from the Wikipedia article <http://en.wikipedia.org/wiki/Polynomial_interpolation>. It is possible to prove that, given a set of n + 1 distinct points, there is exactly one polynomial of degree less than n that passes through every point, but there are infinitely many polynomials of degree higher than n that do as well.

**Differentiated Instruction (Enrichment)**

You might construct a cubic or quartic equation for students in the form of a table of values and ask them to find first differences, second differences, third differences, etc., until they find a difference that is constant. If the nth difference in a table is constant, that means that it is possible to find a polynomial of degree n that goes through all of the points.

At the conclusion of Activity 1.3.1, students may complete **Exit Slip 1.3**.

To introduce **Activity 1.3.2: The absolute value function**, ask students: Is there a difference between a temperature of 10 degrees Celsius and a temperature of -10 degrees Celsius? Is there a difference between having a balance in your checking account of $100 or -$100? Now, is there a difference between driving 100 miles from Hartford to Boston and driving 100 miles from Boston to Hartford? If we only care about the distance on the speedometer of our car, the answer is no. Sometimes in mathematics we are not concerned about the sign of a number, but only its size. The mathematical function that does this for us is called the absolute value function. Define this function for the class by saying the absolute value of x, |x|, is the size of x if we ignore the sign of the number.

Activity 1.3.2 asks students to investigate the following problem: Suppose a business wants to build a factory somewhere in between a number of cities. Where should it be built in order to minimize the total distance from the cities to the factory? It turns out if we simplify the problem so that the cities are all located along a straight line, the solution will depend on the absolute value function. (See the articles Kidd & Pagni (2009) and Stupel (2013).) For two cities, the factory can be located at any point between them. What happens if the number of cities changes to three or four or five? In general, with an odd number of cities, the total distance is minimized when building the factory in the “middle city”; with an even number of cities, the factory can be built anywhere between the middle two cities.

**Differentiated Instruction (Enrichment)**

Finding the point that minimizes the distance between a set of points when the points do not lie on a line is a very difficult problem. The name of the point that minimizes the distance is called the geometric median of the set of points, and it is a mathematical generalization of the median of a set of numbers. When there are 3 or 4 points in a set, it is possible to find the geometric median directly. There is no formula to find a geometric median, though there are some ways known to find an approximation to it. You might want to refer to students to the Wikipedia article on the geometric median (<http://en.wikipedia.org/wiki/Geometric_median>) and ask them to draw what a solution would look like for 3 or 4 points (the method for doing so is described in the article.

To introduce **Activity 1.3.3: Piecewise and step functions**, ask students to pull out a piece of scrap paper (or distribute some scrap paper) and have them write down their best current answer to the question “What is a function?” Read the responses aloud in class. When done, ask the students if any of the responses say anything about a function’s graph being drawn in “one piece.” In order for a function to be a function, do you have to draw its graph without lifting your pencil, or is it OK for it to “jump around?” If we read the definition of a function carefully, does the definition say anything about graphs of functions being in “one piece?” Students should see that the answer is no. In Activity 1.3.3, students will investigate functions that are defined piecewise. This Activity might be started in class and completed as a homework assignment.

**Activity 1.3.4 Symmetry Review** provides a review of two types of symmetry from Geometry, line symmetry and rotational symmetry. The context for these types of symmetry is figures, including a range of corporate logos. This Activity serves as a bridge to considering the types of symmetry a function might exhibit, and could be done in class as a review just before Activity 1.3.5 or used as a homework assignment.

Ask the class “can the graphs of functions also have symmetry?” Is it possible for a function to have rotational symmetry? If we consider the origin as the “center” of a graph, then rotational symmetry means symmetry with respect to rotation around the origin. Is it possible for a function to have line symmetry? How about symmetry across the y-axis? How about symmetry across the x-axis? Students should be able to see that a graph of a function can be symmetric across the y-axis but not across the x-axis. **Activity 1.3.5 Even and Odd Symmetry** asks students to investigate the symmetry of functions.

At the conclusion of Activity 1.3.5, be sure to point out to students the content in the text box at the end of the Activity. While the terms even and odd functions come from functions with even and odd exponents, there are other types of functions that are even and odd. It is not the exponents that determine whether a function is even or odd, it is the property that f(-x) = f(x) or f(-x) = -f(x) that defines even and odd functions. Students will see new types of function in Unit 6, Trigonometric Functions, that are also even and odd.

Also at the end of Activity 1.3.5, students can complete the **Journal Entry** on types of functions.

**Journal** **Entry** **1** In Investigation 3, you have studied a number of different types of functions, including linear, quadratic, exponential, absolute value, and piecewise functions. Which of these have even symmetry? Which of them have odd symmetry? Does each type of function always have the same type of symmetry, or only sometimes?

[Possible answers to this question: quadratic and absolute value functions have even symmetry if they are not “moved” side-to-side. Linear functions have odd symmetry if they pass through the origin. Exponential functions do not have even or odd symmetry. Most piecewise functions do not have even or odd symmetry, though it is possible in some cases.]

**Closure Notes**

Point out to students that already in this course, we have run across a number of different types of functions: linear, quadratic, exponential, absolute value, piecewise, step, and other functions. We have also investigated how functions grow and what kinds of symmetry they have. This way of looking at functions is new, and it is very important. One of the major themes of Algebra 2 is that different types of functions have different characteristics. The ways of studying functions introduced in Investigation 3 will be used throughout Algebra 2 and beyond, to Precalculus and Calculus.

**Vocabulary**

Absolute value function

Even function

Exponential function

Floor function

Greatest integer function

Line symmetry

Linear growth

Nonlinear growth

Odd function

Piecewise function

Quadratic function

Rotational symmetry

Slope

Step function

Symmetric with respect to the origin

Symmetric with respect to the y-axis

**Resources and Materials**

**All students should complete Activities 1.3.1, 1.3.2, 1.3.3, and 1.3.5 in this Investigation. Activity 1.3.4 is a review of symmetry from Geometry and could be skipped if review is not needed.**

Activity 1.3.1 Linear and Nonlinear Growth

Activity 1.3.2 The Absolute Value Function

Activity 1.3.3 Piecewise and Step Functions

Activity 1.3.4 Symmetry Review

Activity 1.3.5 Even and Odd Functions

Graphing calculator/computer software with a graphing utility for all activities

Graph paper for all activities

**References:**

Kidd, M., and Pagni, D. (2009). Investigating absolute value: A real world application, *Australian senior mathematics journal*, 23 (1), 20-30.

Stupel, M. (2013). A special application of absolute value techniques in authentic problem solving. *International journal of mathematical education in science and technology*, 44, 587-595.

<https://www.youtube.com/watch?v=DjlEJNfsOKc>

<http://en.wikipedia.org/wiki/Geometric_median>

<http://en.wikipedia.org/wiki/Polynomial_interpolation>