Smarter Balanced Assessment System: Connecting the Mathematics Claims to Classroom Instruction Grades 6-12

Participant Guide
Connecticut State Department of Education
Fall 2014

Smarter Balanced Assessment Consortium

CONNECTICUT STATE DEPARTMENT OF EDUCATION

Handout 1: Smarter Balanced Assessment System:
Connecting the Mathematics Claims to Classroom Instruction

ACHIEVE THE CORE

Common Core State Standards Shifts in Mathematics

1. **Focus** strongly where the Standards focus	**Focus:** The Standards call for a greater focus in mathematics. Rather than racing to cover topics in a mile-wide, inch-deep curriculum, the Standards require us to significantly narrow and deepen the way time and energy is spent in the math classroom. We focus deeply on the major work* of each grade so that students can gain strong foundations: solid conceptual understanding, a high degree of procedural skill and fluency, and the ability to apply the math they know to solve problems inside and outside the math classroom.
2. **Coherence: think** across grades, and **link** to major topics within grades	**Thinking across grades:** The Standards are designed around coherent progressions from grade to grade. Learning is carefully connected across grades so that students can build new understanding onto foundations built in previous years. Each standard is not a new event, but an extension of previous learning. **Linking to major topics:** Instead of allowing additional or supporting topics to detract from the focus of the grade, these concepts serve the grade level focus. For example, instead of data displays as an end in themselves, they are an opportunity to do grade-level word problems.
3. **Rigor**: in major topics* pursue: • **conceptual understanding**, • procedural skill and **fluency**, and • **application** with equal intensity.	**Conceptual understanding:** The Standards call for conceptual understanding of key concepts, such as place value and ratios. Students must be able to access concepts from a number of perspectives so that they are able to see math as more than a set of mnemonics or discrete procedures. **Procedural skill and fluency:** The Standards call for speed and accuracy in calculation. Students are given opportunities to practice core functions such as single-digit multiplication so that they have access to more complex concepts and procedures. **Application:** The Standards call for students to use math flexibly for applications in problem-solving contexts. In content areas outside of math, particularly science, students are given the opportunity to use math to make meaning of and access content.

High-level Summary of Major Work in Grades K–8

K–2	Addition and subtraction—concepts, skills, and problem solving; place value
3–5	Multiplication and division of whole numbers and fractions—concepts, skills, and problem solving
6	Ratios and proportional relationships; early expressions and equations
7	Ratios and proportional relationships; arithmetic of rational numbers
8	Linear algebra and linear functions

*For a list of major, additional and supporting clusters by grade, please refer to 'Focus in Math' at achievethecore.org/focus pp. 4–12

Handout 4: Smarter Balanced Assessment System: Connecting the Mathematical Claims to Classroom Instruction

A "Snapshot" of the Cognitive Rigor Matrix (Hess, Carlock, Jones, & Walkup, 2009)

Depth of Thinking (Webb) + Type of Thinking (Revised Bloom)	DOK Level 1 Recall & Reproduction	DOK Level 2 Basic Skills & Concepts	DOK Level 3 Strategic Thinking & Reasoning	DOK Level 4 Extended Thinking
Remember	- Recall conversions, terms, facts			
Understand	-Evaluate an expression -Locate points on a grid or number on number line -Solve a one-step problem -Represent math relationships in words, pictures, or symbols	- Specify, explain relationships -Make basic inferences or logical predictions from data/observations -Use models /diagrams to explain concepts -Make and explain estimates	-Use concepts to solve non-routine problems -Use supporting evidence to justify conjectures, generalize, or connect ideas -Explain reasoning when more than one response is possible -Explain phenomena in terms of concepts	-Relate mathematical concepts to other content areas, other domains -Develop generalizations of the results obtained and the strategies used and apply them to new problem situations
Apply	-Follow simple procedures -Calculate, measure, apply a rule (e.g., rounding) -Apply algorithm or formula -Solve linear equations -Make conversions	-Select a procedure and perform it -Solve routine problem applying multiple concepts or decision points -Retrieve information to solve a problem -Translate between representations	-Design investigation for a specific purpose or research question - Use reasoning, planning, and supporting evidence -Translate between problem & symbolic notation when not a direct translation	-Initiate, design, and conduct a project that specifies a problem, identifies solution paths, solves the problem, and reports results
Analyze	-Retrieve information from a table or graph to answer a question -Identify a pattern/trend	-Categorize data, figures -Organize, order data -Select appropriate graph and organize & display data -Interpret data from a simple graph -Extend a pattern	-Compare information within or across data sets or texts -Analyze and draw conclusions from data, citing evidence -Generalize a pattern -Interpret data from complex graph	-Analyze multiple sources of evidence or data sets
Evaluate			-Cite evidence and develop a logical argument -Compare/contrast solution methods -Verify reasonableness	-Apply understanding in a novel way, provide argument or justification for the new application
Create	- Brainstorm ideas, concepts, problems, or perspectives related to a topic or concept	-Generate conjectures or hypotheses based on observations or prior knowledge and experience	-Develop an alternative solution -Synthesize information within one data set	-Synthesize information across multiple sources or data sets -Design a model to inform and solve a practical or abstract situation

Illustrative Mathematics

8.NS Comparing Rational and Irrational Numbers

Alignments to Content Standards

- **Alignment: 8.NS.A.2**

Tags

- *This task is not yet tagged.*

For each pair of numbers, decide which is larger without using a calculator. Explain your choices.

a. π^2 or 9

b. $\sqrt{50}$ or $\sqrt{51}$

c. $\sqrt{50}$ or 8

d. -2π or -6

Commentary

This task can be used to either build or assess initial understandings related to rational approximations of irrational numbers.

Solutions

Solution: Solution

a. $\pi > 3$ so $\pi^2 > 9$.

b. $\sqrt{50} < \sqrt{51}$ because $50 = (\sqrt{50})^2 < (\sqrt{51})^2 = 51$.

c. $7^2 = 49$ and $8^2 = 64$. Thus we have that $\sqrt{49} < \sqrt{50} < \sqrt{64}$. So $\sqrt{50} < 8$.

d. $\pi > 3$ so $2\pi > 2 \cdot 3$. If you look at these numbers on the number line, that means that 2π is farther to the right than 6. When you look at their opposites, -2π will be farther to the left than -6, so $-2\pi < -6$.

HANDOUT #6

Smarter Balanced Assessment System: Connecting the Mathematics Claims to Classroom Instruction

Bringing It Back to the Standards and the Teaching Practices

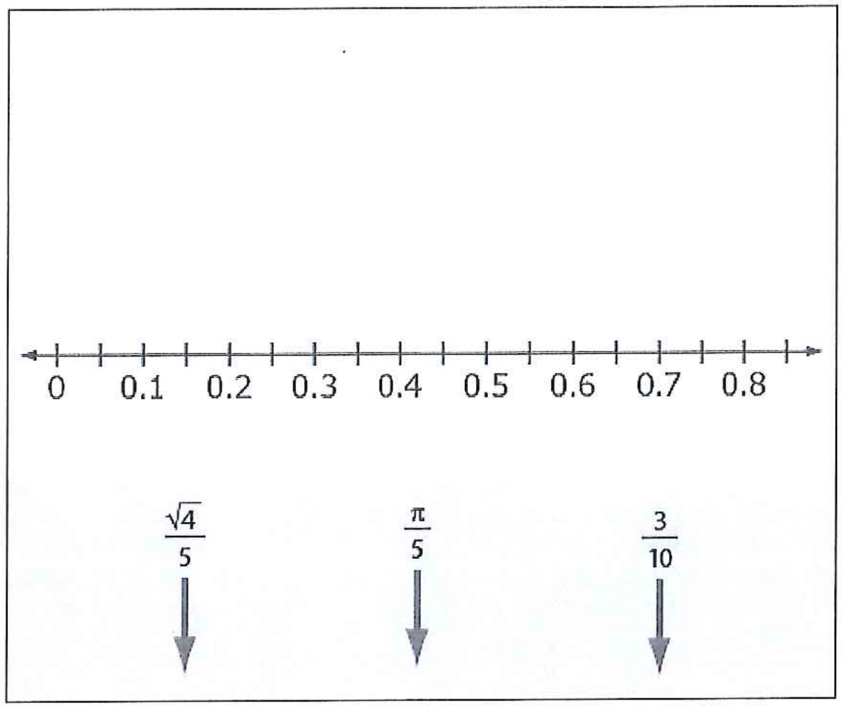
Standards for Mathematical Content Domain Cluster Heading Standard(s)	Standards for Mathematical Practice	Mathematics Teaching Practices	
		Teacher Actions	Student Actions

Grade 8 Mathematics

Item	Claim	Domain	Target	DOK	CONTENT	MP	Key
#1	1	NS	A	2	8.NS.A.2	6	See exemplar

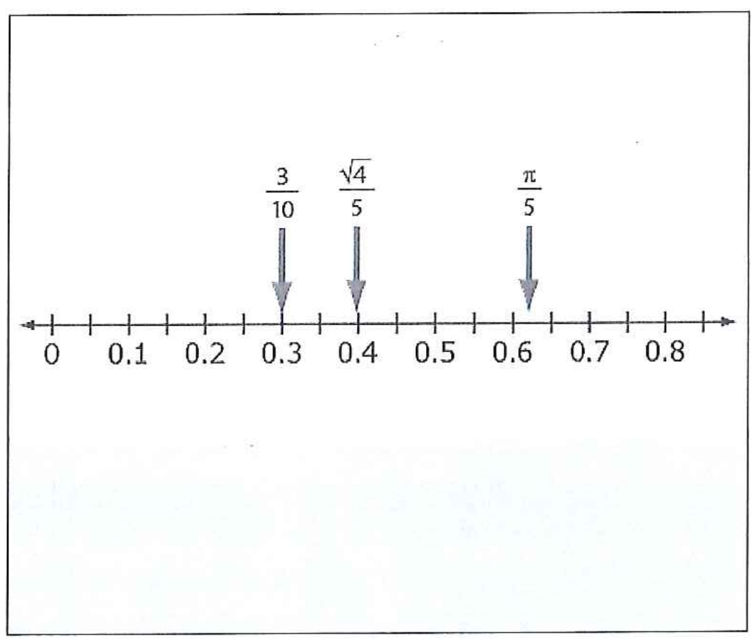
1860

Drag each number to its correct position on the number line.



Exemplar: (shown at right)

- $\frac{3}{10}$ at 0.3
- $\frac{\sqrt{4}}{5}$ at 0.4
- $\frac{\pi}{5}$ just to the right of 0.6



Rubric:
(2 points) Student places all three values at the correct positions on the number line.

(1 point) Student places two of the three values at the correct positions on the number line.

Illustrative Mathematics

A-SSE Graphs of Quadratic Functions

Alignments to Content Standards

- **Alignment: A-SSE.B.3**
- **Alignment: F-IF.C.7**

Tags

- *This task is not yet tagged.*

a. Graph these equations on your graphing calculator at the same time. What happens? Why?

$$y_1 = (x-3)(x+1)$$
$$y_2 = x^2 - 2x - 3$$
$$y_3 = (x-1)^2 - 4$$
$$y_4 = x^2 - 2x + 1$$

b. Below are the first three equations from the previous problem.

$$y_1 = (x-3)(x+1)$$
$$y_2 = x^2 - 2x - 3$$
$$y_3 = (x-1)^2 - 4$$

These three equations all describe the same function. What are the coordinates of the following points on the graph of the function? From which equation is each point most easily determined? Explain.

i. vertex: _____

ii. y-intercept: _____

iii. x-intercept(s): _____

c. Make up an equation for a quadratic function whose graph satisfies the given condition. Use whatever form is most convenient.

i. Has a vertex at $(-2,-5)$

ii. Has a y-intercept at $(0,6)$

iii. Has x-intercepts at $(3,0)$ and $(5,0)$

iv. Has x-intercepts at the origin and $(4,0)$

v. Goes through the points $(4,2)$ and $(1,2)$

Commentary

This exploration can be done in class near the beginning of a unit on graphing parabolas. Students need to be familiar with intercepts, and need to know what the vertex is. It is effective after students have graphed parabolas in vertex form ($y = a(x-h)^2 + k$), but have not yet explored graphing other forms. Part (a) is not obvious to them; they are excited to realize that equivalent expressions produce the same graph. Parts (b) and (c) lead to important discussions about the value of different forms of equations, culminating in a discussion of how we can convert between forms and when we might want to do so.

A natural extension of this task is to have the students share some of the different equations that they found for a given condition and have them graph two or more simultaneously. For example, students could graph three different equations that all have the same x-intercepts and discuss the effect that the different constant factors have on the graph.

Solutions

Solution: Solutions

a. When you graph these four equations, only two different parabolas are shown. This is because the first three equations are equivalent, and so all produce the same graph. We can see the equivalence as follows:

- If we multiply the factors given in the first equation, we'll get the second equation:

$$(x-3)(x+1) =$$
$$x^2 - 3x + 1x - 3 =$$
$$x^2 - 2x - 3$$

- Similarly, if we multiply out the perfect square and combine like terms in the third equation, we also get the second one:

$$(x-1)^2 - 4 =$$
$$x^2 - 2x + 1 - 4 =$$
$$x^2 - 2x - 3$$

The fourth function produces a different graph. We can see that the difference between it and y_2 is just 4, so that graph is 4 units below the other one.

b. i. The vertex is $(1, -4)$ which is most visible in y_3 since the vertex occurs at the point where the squared portion is zero.

ii. The y-intercept is $(0, -3)$, which is visible as the constant in y_2 since the other terms are 0 when you plug in $x = 0$.

iii. The x-intercepts are $(3, 0)$ and $(-1, 0)$, which are most visible in y_1 since you can find the roots of the polynomial using the zerofactor property and thus the intercepts correspond to the zeros of each factor.

c. Note: each of these problems has many possible answers. We're including three possible answers for each one, to demonstrate the type of variability you might expect to see in a class. Asking students for three possible answers is a great extension for students - it gets them thinking about the effects of the different parts of the equation.

i. The following have a vertex at $(-2, -5)$:

$$y = (x+2)^2 - 5$$
$$y = -(x+2)^2 - 5$$
$$y = 3(x+2)^2 - 5$$

ii. The following have a y-intercept of $(0, -6)$:

$$y = x^2 - 6$$
$$y = x^2 + 13x - 6$$
$$y = 2x^2 - 6$$

iii. The following have x-intercepts of $(3, 0)$ and $(5, 0)$:

$$y = (x-3)(x-5)$$
$$y = 2(x-3)(x-5)$$
$$y = -7(x-3)(x-5)$$

iv. The following have x-intercepts at the origin and $(-4, 0)$:

$$y = x(x+4)$$
$$y = 12x(x+4)$$
$$y = -x(x+4)$$

v. The following go through the points $(-4, 2)$ and $(1, 2)$:

$$y = (x+4)(x-1) + 2$$
$$y = 2(x+4)(x-1) + 2$$
$$y = -\frac{1}{2}(x+4)(x-1) + 2$$

(Note: students will likely need to experiment quite a bit to find an equation that satisfies these constraints.)

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Standards for Mathematical Content Domain Cluster Heading Standard(s)	Standards for Mathematical Practice	Mathematics Teaching Practices	
		Teacher Actions	Student Actions

Item	Claim	Domain	Target	DOK	CONTENT	MP	Key
#3	1	A-SSE	D	1	HSA.SSE.A.2	7	See exemplar

1915

Determine whether each expression is equivalent to $(x^3 + 8)$. Select Yes or No for each expression.

	Yes	No
$(x + 8)^3$	☐	☐
$(x - 2)(x^2 + 2x + 4)$	☐	☐
$(x + 2)(x^2 - 2x + 4)$	☐	☐

Exemplar: (shown at right)

	Yes	No
$(x + 8)^3$	☐	☑
$(x - 2)(x^2 + 2x + 4)$	☐	☑
$(x + 2)(x^2 - 2x + 4)$	☑	☐

Rubric: (1 point) Student correctly classifies the expressions (NNY).

Illustrative Mathematics

6.EE Seven to the What?!?

Alignments to Content Standards

- **Alignment: 6.EE.A.1**

Tags

Tags: AMC

a. **What is the last digit of 7^{2011}? Explain.**

b. **What are the last *two* digits of 7^{2011}? Explain.**

Commentary

At first glance, this might seem like an impossible problem because the number 7^{2011} is much too large to evaluate even with a calculator. It requires a starting point, and, to many, such a starting point might not be obvious. Students who successfully answer this problem must "look for and express regularity in repeated reasoning" (MP8), and so the purpose of this task is to give students an opportunity to engage in MP8 and to practice working with positive integer exponents. There are two key steps involved in the solving this problem:

- **Recognizing (or at least presuming) that the last two digits of 7^{2011} can be found without calculating the number 7^{2011}.**
- **Identifying a pattern in the last two digits of successive powers of 7 (which hopefully appears quickly!).**

Once students see that there is a pattern to the last two digits of successive powers of 7, they can easily guess the correct answer. A complete explanation for why we know the pattern will continue will probably not be found by a 6th grader, but because it is based on place value, it could be understood by sixth graders who have a good understanding of the base-ten number system.

It would be fun to replace the exponent 2011 with the current year, which doesn't change the substance of the problem in any way. This task is only intended for instructional purposes and would be completely inappropriate for high-stakes assessment (in case anyone wonders).

This task was adapted from problem #22 on the 2011 American Mathematics Competition (AMC) 8 Test. For the 2011 AMC 8, which was taken by 153,485 students, the multiple choice answers for the problem which asked students to find the ten's digit of 7^{2011} had the following had the following distribution:

Choice	Answer	Percentage of Answers
(A)	0	16
(B)	1	16
(C)	3	14
(D)*	4	21
(E)	7	21
Omit	-	12

Of the 153485 students: 72,648 (47%) were in 8th grade, 50,433 (33%) were in 7th grade, and the remainder were less than 7th grade.

Solutions

Solution: 1 Making a Table

Below is a table listing the first 8 powers of 7 along with the final digits of these powers:

Power of 7	**Result**	**Last two digits**
7^1	7	07
7^2	49	49
7^3	343	43
7^4	2401	01
7^5	16,807	07
7^6	117,649	49
7^7	823,543	43
7^8	5,764,801	01

Notice that the last two digits are following a simple repeating pattern:

$$07, 49, 43, 01.$$

We are interested in the 2011th number in this sequence. The 2000th power will go through this sequence of 4 numbers $2000 \div 4 = 500$ times. The 2008th power will go through this sequence of 4 numbers 2 more times. Thus 7^{2009} will end in 07, 7^{2010} ends in 49 and 7^{2011} ends in 43. So we have an answer to both questions:

a. **The ones place of 7^{2011} is a 3.**

b. **The tens place of 7^{2011} is a 4.**

Solution: 2 Reasoning with place value

a. To find successive powers of 7, we can multiply by 7 repeatedly, with 2011 factors of 7 in all to get 7^{2011}. We are only interested in the ones place and start with 7^3 as an example:

$$\begin{aligned} 7^3 &= 7 \times 7^2 \\ &= 7 \times 49 \\ &= 7 \times (4 \times 10 + 9) \\ &= (7 \times 4 \times 10) + (7 \times 9). \end{aligned}$$

Since $7 \times 9 = 63$, the ones place of 7^3 will be 3. Notice that the 4 tens in 49 do not play a role in determining the ones place of the product. This makes sense as these tens will only influence the higher place values in the product. So when we are looking to find the ones digit of 7^{2011} we only need to pay attention to the ones digit in each successive power. Starting with the first power these are 7, 9, 3, 1 and then the sequence repeats. Since

$$2011 = 502 \times 4 + 3$$

this means that for 7^{2011} we will go completely through this sequence of last digits 502 times and then land on the third one which is 3. So the last digit of 7^{2011} is 3.

b. We can apply the same reasoning to find the last two digits of 7^{2011}. Taking 7^4 as an example, note that

$$\begin{aligned} 7^4 &= 7 \times 7^3 \\ &= 7 \times 343 \\ &= 7 \times (3 \times 100 + 43) \\ &= (7 \times 3 \times 100) + (7 \times 43). \end{aligned}$$

As above, the only part of this expression which contributes to the tens and ones digits is 7×43. We have $7 \times 43 = 301$ and so the digit in the hundreds place of 7^4 is 0 and the digit in the ones place of 7^4 is 1. At this point we enter a nice pattern as we see by studying at $7^5 = 7 \times 7^4$: since we know 7^4 has a zero in the hundreds place and a 1 in the ones place, when we multiply by 7 we will get a 0 in the hundreds place and a 7 in the ones place. So we will be back to where we started with the same tens digit and ones digit as 7^1. This pattern will continue so the first, fifth, ninth (and so on) powers of 7 end in 07. Since

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Smarter Balanced Assessment Consortium

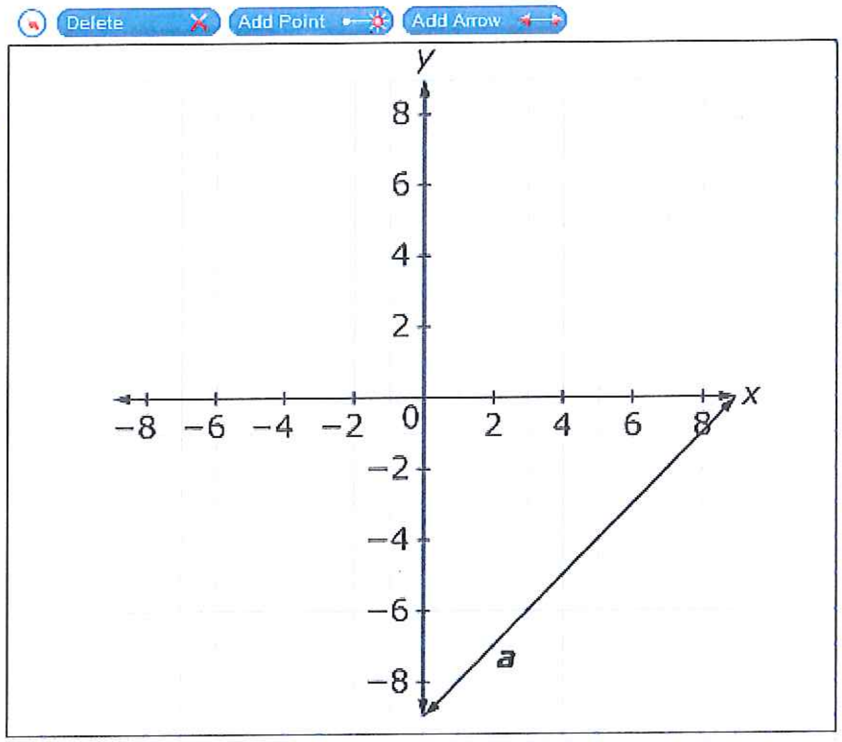
Grade 8 Mathematics

Item	Claim	Domain	Target	DOK	CONTENT	MP	Key
#22	2	EE	D, B	2	8.EE.C.8	1	See exemplar

1834

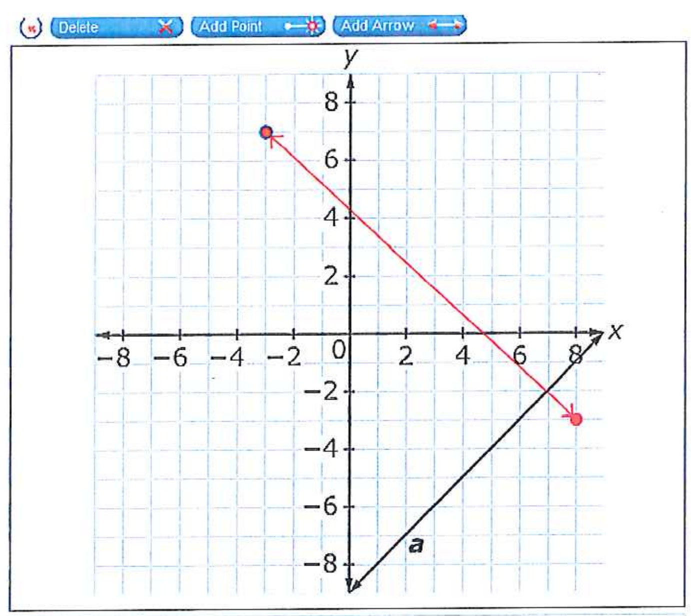
Line *a* is shown on the graph. Use the Add Arrow tool to construct line *b* on the graph so that:

- Line *a* and line *b* represent a system of linear equations with a solution of $(7, -2)$.
- The slope of line *b* is greater than -1 and less than 0.
- The *y*-intercept of line *b* is positive.



Exemplar: (shown at right)
Other correct graphs are possible.

Rubric: (1 point) Student constructs a line that meets the stated requirements.



Illustrative Mathematics

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Item	Claim	Domain	Target	DOK	CONTENT	MP	Key
#23	2	S-CP	A	2	HSS.CP.A	1, 2, 6	See exemplar

1927

At a local fair, the price of admission includes the opportunity for a person to spin a wheel for free ride tickets.

- Each spin of the wheel is a random event.
- The result from each spin of the wheel is independent of the results of previous spins.
- Each spin of the wheel awards tickets according to the probabilities shown below.

Spin the Wheel

1 ticket	35%
2 tickets	25%
3 tickets	20%
5 tickets	15%
10 tickets	5%

Let X be the number of tickets a person wins based on 2 spins. There are 13 possible values for X.

Some values of X are more common than others. For example, winning only 2 tickets in 2 spins is a somewhat common occurrence with probability 0.1225. It means the person wins 1 ticket on the first spin and 1 ticket on the second spin (0.35 • 0.35). A list of the possible values of X and the corresponding probabilities for most values of X is shown below.

Fill in the three missing probability values in the table.

X	Probability
2	0.1225
3	0.1750
4	
5	0.1000
6	0.1450
7	0.0750
8	0.0600
10	
11	0.0350
12	0.0250
13	
15	0.0150
20	0.0025

(Item #23 continued)

Exemplar:

X	Probability
2	0.1225
3	0.1750
4	0.2025
5	0.1000
6	0.1450
7	0.0750
8	0.0600
10	0.0225
11	0.0350
12	0.0250
13	0.02
15	0.0150
20	0.0025

Rubric:
(3 points) Student enters three correct probabilities.
(2 points) Student enters two out of three correct probabilities.
(1 point) Student enters one out of three correct probabilities.

Illustrative Mathematics

7.RP Dueling Candidates

Alignments to Content Standards

- **Alignment: 7.RP.A**

Tags

- *This task is not yet tagged.*

Joel and Marisa are running for president at their middle school (grades 6-8). After the votes are in, Joel and Marisa are each convinced that they have won the election:

- **Joel argues that he has won a larger percentage of the overall vote than Marisa so he should be the new president.**
- **Marisa argues that she has won a larger percentage than Joel of the 6th grade vote and the 7th grade vote. Since the majority of the grades voted for her, she should be the new president.**

Is it possible that both Joel and Marisa are making accurate claims? Explain.

Commentary

The goal of this task is to have students examine some properties of ratios (and fractions) in an important real world context. Students will gain practice working with ratios while investigating some of the complexities of voting theory. This task can be made less open-ended by supplying, for example, the total number of students at the school or even the number of students in each class: the teacher may also wish to discuss the analysis in the first paragraph of the first solution if students are stuck. As written, it is intended to be an engaging, open-ended question and will require ample time. It is an ideal task for group work.

If an extension of the problem is desired, the teacher might ask if it is possible for Joel to win a larger percentage of the overall vote than Marisa while Marisa wins a larger percentage of the vote within *every* grade level. This scenario is impossible because it would mean that Marisa wins more votes than Joel in each grade but fewer votes when the grades are added up.

It is important for students to understand that both Joel and Marisa have legitimate arguments and so it is essential that the rules governing the election be specified in advance. For a further discussion, the teacher may wish to show students numbers for the 2000 presidential election:

http://en.wikipedia.org/wiki/United_States_presidential_election,_2000

Al Gore received *more* overall votes than George W. Bush (Joel's argument) but Bush won the election because presidential elections are determined on a state by state basis (Marisa's argument). The rules governing presidential elections follow Marisa's line of reasoning (although there are additonal weights involved as each state has a given number of delegates), where the states play the role of the different grade levels at the school. So George W. Bush was elected president even though he received fewer votes than Al Gore.

Students working on this task will engage in MP2, Reason Abstractly and Quantitatively, as the main work of the task involves constructing and reasoning with numbers which satisfy constraints (which also must be reasoned out from the context). The task also provides an opportunity to work on MP3, Construct Viable Arguments and Critique the Reasoning of Others. This may happen at two levels: first students will critique the supplied reasoning of Joel and Marisa and, secondly, they may well disagree about which line of reasoning is more convincing and then they will examine and critique the reasoning of one another.

This task was designed for an NSF supported summer program for teachers and undergraduate students held at the University of New Mexico from July 29 through August 2, 2013 (http://www.math.unm.edu/mctp/).

Solutions

Solution: 1 Working with Fractions and Percents (6.RP.3)

We are given that Joel has won the majority of the total number of votes at the school. On the other hand, when the vote is divided up by grade level, Marisa has a higher percentage of the 6th grade and 7th grade votes. Before choosing numbers, note that if Joel wins a higher percentage of the 8th grade votes than Marisa then he has a chance to make up for the ground he lost in the 6th and 7th grade. To see if the two scenarios are consistent we have to choose numbers so that Joel makes up more ground on the 8th grade votes than he lost in the combined 6th and 7th grade votes.

We will assume for simplicity that there are 600 students at the middle school. We will also assume that all 600 students vote for either Joel or Marisa. We need to divide the 600 students between the three grades and begin with the assumption that there are 200 in each grade. As observed above, we need to make sure that Joel wins the 8th grade vote by more than Marisa's combined 6th and 7th grade wins. Suppose the votes go as in the table below:

Candidate	**6th grade votes (% of 6th grade votes)**	**7th grade votes (% of 7th grade votes)**	**8th grade votes (% of 8th grade votes)**
Joel	80 (40%)	90 (45%)	140 (70%)
Marisa	120 (60%)	110 (55%)	60 (30%)

We can see that Marisa won more of the 6th and 7th grade votes than Joel and so she won a larger percentage of the 6th grade votes and the 7th grade votes. For the overall vote, however, Joel received $80 + 90 + 140 = 310$ votes while Marisa received $120 + 110 + 60 = 290$ votes. Converting to percentages, we find that Joel has won

$$100 \times \frac{310}{(600)} \approx 52\%$$

of the vote while Marisa has won

$$100 \times \frac{290}{(600)} \approx 48\%$$

of the vote. So Joel has won a larger percentage of the overall vote than Marisa.

Solution: 2 Working with ratios (7.RP.3)

We begin with a table representing the scenario described in the problem with question marks where we need to provide information:

	6th grade vote	**7th grade vote**	**8th grade vote**	**total school vote**
Marisa:Joel	?	?	?	?

For the 6th and 7th grade columns, we know that Marisa has won a larger percentage of those votes than Joel. For the 8th grade and total school columns, Joel has won a larger percentage of those votes than Marisa. We need to see if it is possible to assign numbers so that all 4 of these conditions hold. We begin by setting the size of each class: for simplicity, we will assume that there are 100 students in each of the 6th, 7th, and 8th grade classes and so there are 300 total students.

For the first two columns we start by putting in some numbers where the vote favors Marisa and then see if we can complete the table:

	6th grade vote	7th grade vote	8th grade vote	total school vote
Marisa:Joel	55:45	60:40	?	?

Here Marisa has won a larger percentage of the 6th grade vote (55 percent to 45 percent) and a larger percentage of the 7th grade vote (60 percent to 40 percent). Note that if Marisa had won the 6th and 7th grade votes by a margin of more than 100 votes, then it is impossible for Joel to make up the difference with the 100 votes in the 8th grade. With the values we have entered for the 6th and 7th grade, Marisa has won these two by a combined margin of 30 votes. So for Joel to win the 8th grade vote *and* the overall vote, he needs a margin of more than 30 votes in the 8th grade. Here is what we get if we assume a 70 to 30 split of the 8th grade vote in favor of Joel:

	6th grade vote	7th grade vote	8th grade vote	total school vote
Marisa:Joel	55:45	60:40	30:70	145:155

The table above gives values which match the situation described: Marisa has won a larger percentage of the 6th and 7th grade votes. The 8th grade vote splits as 70% for Joel and 30% for Marisa. There are 300 votes total so to find the percentage of the overall vote won by each candidate we can divide each number in the total vote ratio by 3:

$$\frac{145}{3} : \frac{155}{3} :: 48.3 : 51.6.$$

Here we see that for the total vote, Marisa won a little over 48 percent and Joel won a little less than 52 percent.

HANDOUT #6

Smarter Balanced Assessment System: Connecting the Mathematics Claims to Classroom Instruction

Bringing It Back to the Standards and the Teaching Practices

Standards for Mathematical Content Domain Cluster Heading Standard(s)	Standards for Mathematical Practice	Mathematics Teaching Practices	
		Teacher Actions	Student Actions

Item	Claim	Domain	Target	DOK	CONTENT	MP	Key
#11	3	NF	A	3	5.NF.B.4	2, 3, 7	See exemplar

1857

Look at the equation.

$$\frac{2}{3} \times \frac{\square}{\square} = n$$

Sarah claims that for any fraction multiplied by $\frac{2}{3}$, n will be less than $\frac{2}{3}$.

To convince Sarah that this statement is only sometimes true:

Part A: Drag one number into each box so the product, n, is less than $\frac{2}{3}$.

Part B: Drag one number into each box so the product, n, is **not** less than $\frac{2}{3}$.

1
2
3
4
5
6
7
8
9

Delete

Part A: **Product *n* is less than $\frac{2}{3}$**

$$\frac{2}{3} \times \frac{\square}{\square} = n$$

Part B: **Product *n* is not less than $\frac{2}{3}$**

$$\frac{2}{3} \times \frac{\square}{\square} = n$$

Exemplar: (shown at right)

Rubric: (1 point) Student drags one number into each box to create an equation where n is less than $\frac{2}{3}$ in Part A, and drags one number into each box to create an equation to show that Sarah's claim is incorrect in Part B (e.g., Part A $\frac{1}{3}$, Part B $\frac{2}{1}$). This exemplar response represents only one possible solution. Other correct responses are possible.

Part A: **Product *n* is less than $\frac{2}{3}$**

$$\frac{2}{3} \times \frac{1}{3} = n$$

Part B: **Product *n* is not less than $\frac{2}{3}$**

$$\frac{2}{3} \times \frac{2}{1} = n$$

Handout 8b: Smarter Balanced Assessment System:
Connecting the Mathematics Claims to Classroom Instruction

Illustrative Mathematics

A.SSE Taxes and Sales

Alignments to Content Standards

- **Alignment: A-SSE.B**

Tags

Tags: MP 4

Judy is working at a retail store over summer break. A customer buys a $50 shirt that is on sale for 20% off. Judy computes the discount, then adds sales tax of 10%, and tells the customer how much he owes. The customer insists that Judy first add the sales tax and then apply the discount. He is convinced that this way he will save more money because the discount amount will be larger.

a. **Is the customer right?**

b. **Does your answer to part (a) depend on the numbers used or would it work for any percentage discount and any sales tax percentage? Find a convincing argument using algebraic expressions and/or diagrams for this more general scenario.**

Commentary

This task is not about computing the final price of the shirt but about using the structure in the computation to make a general argument. The key underlying idea is that multiplication is commutative, which we often just take for granted and don't feel needs any explanation. In this case, the context of the problem makes it not obvious at all that we can switch the order of the two computations, but it becomes quite obvious after observing that the application of both the discount and the sales tax are just instances of multiplication. Since the order in which we multiply is irrelevant, the answer must be the same regardless of which we apply first.

The solution presents both an algebraic approach to the general result in part (b), and also a diagram that illustrates the same result graphically.

This task presents a good opportunity for students to construct a viable argument and critique the reasoning of others (MP3).

Solutions

Solution: Solution

a. Judy first takes 20% off which gives a new price of \$50(0.8) = \$40. She then adds the 10% sales tax for a final price of \$40(1.1) = \$44. The customer first adds 10% for a new price of \$50(1.1) = \$55. He then takes 20% off for a final price of \$55(0.8) = \$44.

   The customer is right to say that the discount amount will be larger, it is \$11 opposed to \$10 with his method. But the additional \$1 just gets subtracted from the tax amount that was added in the first step. So the final price is the same in both cases.

   It does not matter in which order the discount and tax are computed.

b. If we don't actually perform the computations but just record them we find the following:

   Judy: $50(0.8)(1.1) = 44$

   customer: $50(1.1)(0.8) = 44$

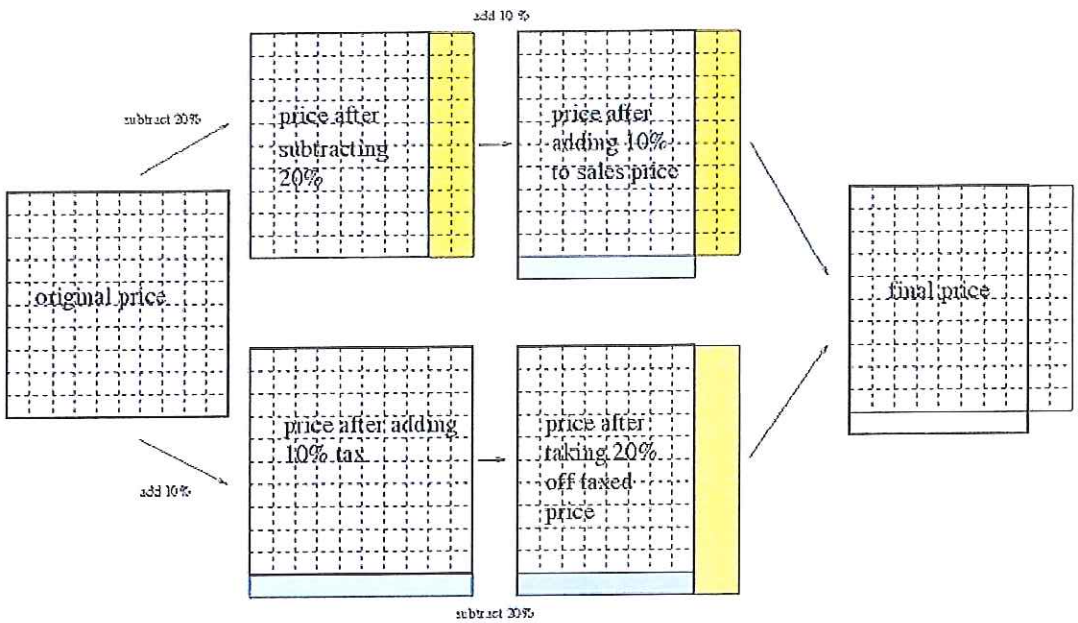
   We see that it is not surprising that both computations get the same answer, since $(0.8)\cdot(1.1) = (1.1)\cdot(0.8)$

   This result will generalize if we replace \$50, 20%, 10% by any other numbers. If we let P stand for the original price, s for the sales percentage and t for the tax percentage, we have

$$P\left(1-\frac{s}{100}\right)\left(1+\frac{t}{100}\right)=P\left(1+\frac{t}{100}\right)\left(1-\frac{s}{100}\right)$$

We see that changing the order in which the sale and the tax are applied does not matter.

We can also visualize this with the following diagram. Yellow represents the action of subtracting 20% and blue represents the action of adding 10%. We see that both paths result in the same final answer. Even though the diagram uses the numbers from the problem, we can see from the structure in the diagram that both paths will result in the same final price even if the yellow and blue areas are altered.



HANDOUT #6

Smarter Balanced Assessment System: Connecting the Mathematics Claims to Classroom Instruction

Bringing It Back to the Standards and the Teaching Practices

Standards for Mathematical Content Domain Cluster Heading Standard(s)	Standards for Mathematical Practice	Mathematics Teaching Practices	
		Teacher Actions	Student Actions

High School Mathematics

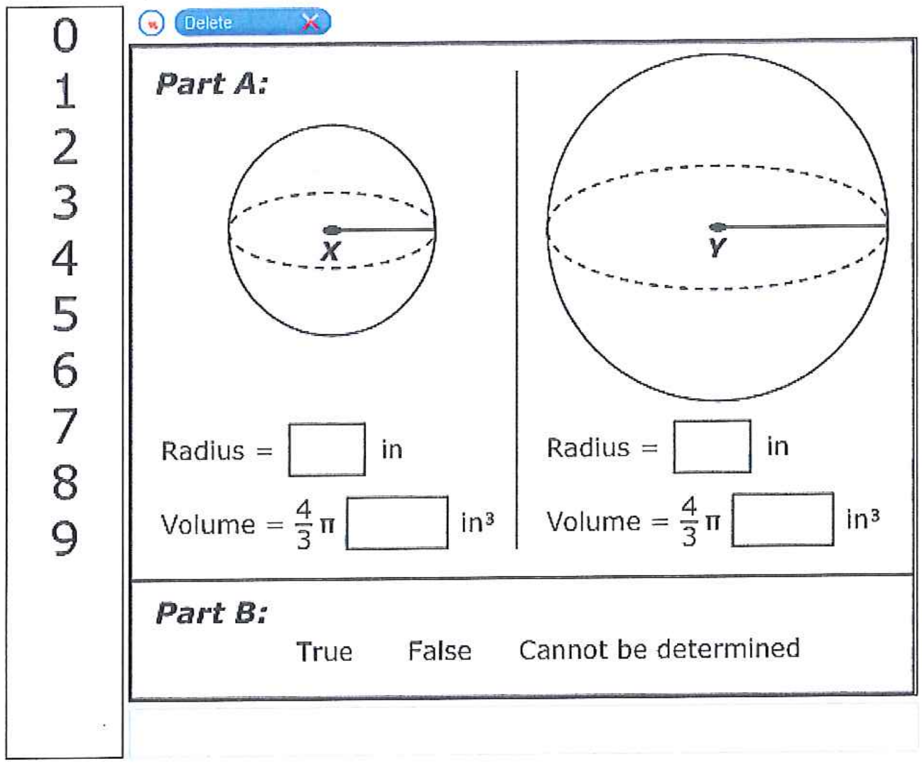
Item	Claim	Domain	Target	DOK	CONTENT	MP	Key
#14	3	G-SRT	A	3	HSG.SRT.A	3	See exemplar

2029

The radius of sphere *Y* is twice the radius of sphere *X*. A student claims that the volume of sphere *Y* must be exactly twice the volume of sphere *X*.

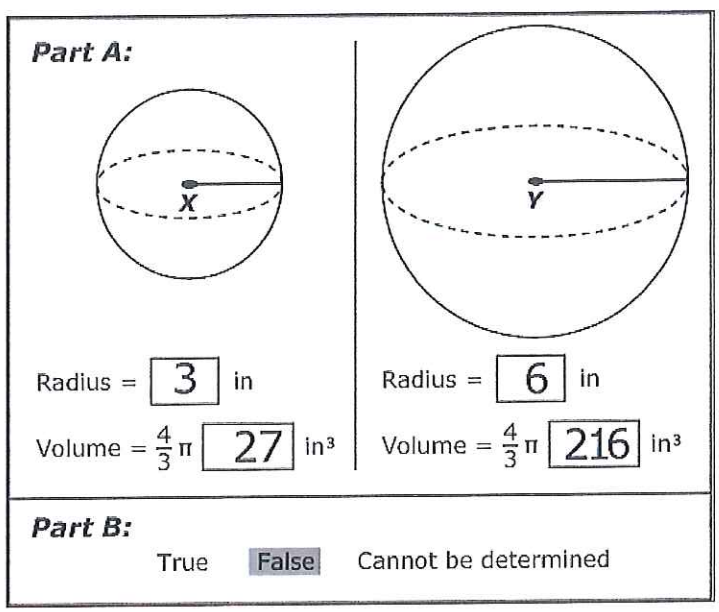
Part A: Drag numbers into the boxes to create one example to evaluate the student's claim.

Part B: Decide whether the student's claim is true, false, or cannot be determined. Select the correct option.



Exemplar: (shown at right)
Other correct examples are possible.

Rubric: (1 point) Student creates a correct example and determines that the claim is false.



Handout 9a: Smarter Balanced Assessment System: Connecting the Mathematics Claims to Classroom Instruction

How does Ms. Walker encourage her students to take responsibility for their learning?
Why does Ms. Walker use feedback questions instead of comments when responding to student work?
What is powerful about grade level teachers looking at student work together?

HANDOUT #6

Smarter Balanced Assessment System: Connecting the Mathematics Claims to Classroom Instruction

Bringing It Back to the Standards and the Teaching Practices

Standards for Mathematical Content Domain Cluster Heading Standard(s)	Standards for Mathematical Practice	Mathematics Teaching Practices	
		Teacher Actions	Student Actions

Item	Claim	Domain	Target	DOK	CONTENT	MP	Key
#25	4	EE	F, G	3	6.EE.B.8	1, 2, 4	See exemplar

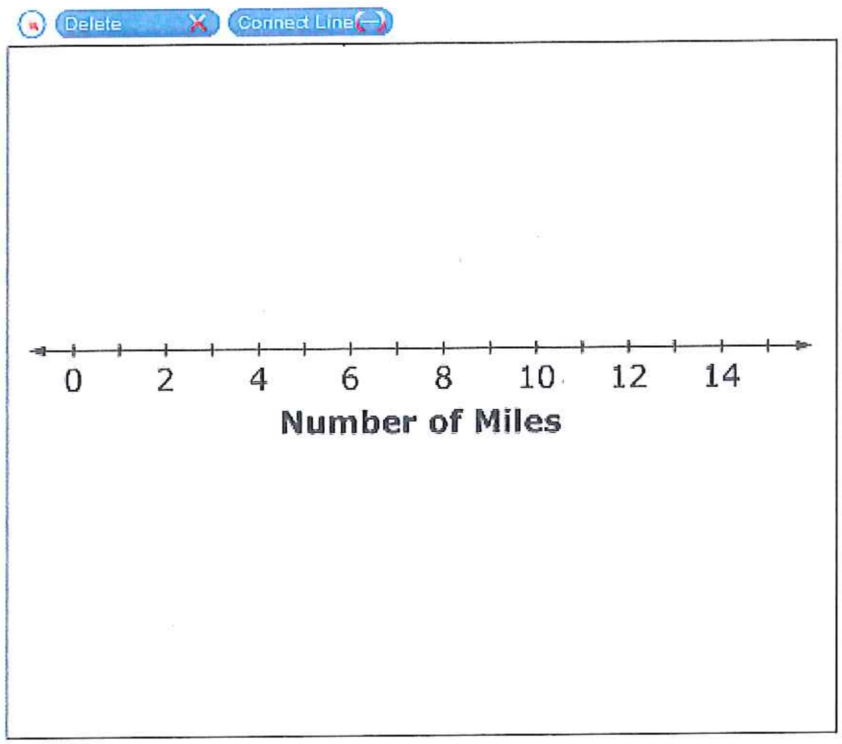
1798

A boat takes 3 hours to reach an island 15 miles away.

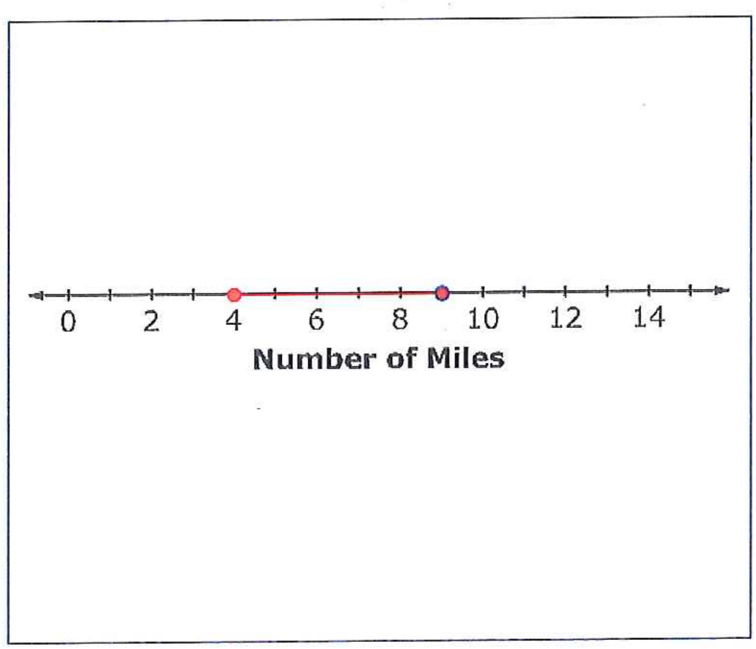
The boat travels:

- at least 1 mile but no more than 6 miles during the first hour
- at least 2 miles during the second hour
- exactly 5 miles during the third hour

Use the Connect Line tool to show the range of miles the boat could have traveled during the **second** hour, given the conditions above.



Exemplar: (shown at right)



Rubric: (1 point) Student draws a segment to indicate the correct range of miles from 4 to 9.

Illustrative Mathematics

A-SSE Course of Antibiotics

Alignments to Content Standards

- **Alignment: A-SSE.B.4**

Tags

Tags: MP 4

Susan has an ear infection. The doctor prescribes a course of antibiotics. Susan is told to take 250 mg doses of the antibiotic regularly every 12 hours for 10 days.

Susan is curious and wants to know how much of the drug will be in her body over the course of the 10 days. She does some research online and finds out that at the end of 12 hours, about 4% of the drug is still in the body.

a. **What quantity of the drug is in the body right after the first dose, the second dose, the third dose, the fourth dose?**

b. **When will the total amount of the antibiotic in Susan's body be the highest? What is that amount?**

c. **Answer Susan's original question: Describe how much of the drug will be in her body at various points over the course of the 10 days.**

Commentary

This task presents a real world application of finite geometric series. The context can lead into several interesting follow-up questions and projects. Many drugs only become effective after the amount in the body builds up to a certain level. This can be modeled very well with geometric series.

The task should be used primarily for instruction, as questions about the context might confuse students attempting the task during an assessment. (For example, the task uses the true hypothesis that the amount of drug continues to decay exponentially, so that 4% of 4% of it persists in the system after 2 hours, but students might need help making that assumption explicit). As an example of classroom use, one could use the task as part of a set of real-life situations to which geometric series apply, or assign the task to cooperative learning groups to further engagement in the mathematical practices that must be brought to bear. In particular, this further engagement could be manifest itself in attention to practice standards, e.g., MP8 (regularity in repeated reasoning is used to ascertain the geometric series as opposed to carrying out the process 19 times) and MP4 (a geometric series is used to model the situation presented by task).

Finally, an opportunity presents itself here to discuss reasonableness of numerical approximations. In the solution to part (b), we encounter numbers like $(.04)^{19}$, which is approximately 3×10^{-27}, a quantity of milligrams so incredibly tiny that it is negligible for all realistic purposes. The solution addresses this only briefly, by noting that it is unreasonable to report back the full scale of decimal accuracy the abstract mathematical model predicts.

Solutions

Solution: Solution

a. Let Q_1 be the amount of the drug in the body after the first dose. Since the dose is 250 mg, $Q_1 = 250$. To find Q_2, the amount of the drug in the body after the second dose, we have to find out how much of the first dose is still present and add the new dose. Since after 12 hours about 4% of the drug is still present in the body we get:

$$Q_2 = 0.04Q_1 + 250 = (0.04)250 + 250 = 260.$$

Right after the second dose, 260 mg of the drug are present in the body. We use the same reasoning to find Q_3 :

$$Q_3 = (0.04)Q_2 + 250 = (0.04)((0.04)250 + 250) + 250 = (0.04)^2 250 + (0.04)250 + 250 = 260.4.$$

After the third dose, 260.4 mg of the drug are present in the body. We can see a pattern emerging and find

$$Q_4 = (0.04)^3 250 + (0.04)^2 250 + (0.04)250 + 250 = 260.416.$$

b. Since every time Susan takes a new dose of the antibiotic, a small part of the previous doses is still present, the total amount of the antibiotic in her body will be the highest right after the last dose, which is the 20th dose. To find out how large that amount is, we have to find Q_{20}.

From computing Q_1, Q_2, Q_3 and Q_4 we saw a pattern of a finite geometric series emerging. So we have

$$\begin{aligned} Q_{20} &= 250 + 250(0.04) + 250(0.04)^2 + \cdots + 250(0.04)^{19} \\ &= 250(1 + 0.04 + 0.04^2 + \cdots + 0.04^{19}) \\ &= 250\frac{1 - 0.04^{20}}{1 - 0.04} \\ &= 260.4167 \end{aligned}$$

So the greatest amount of antibiotics in Susan's body is 260.4167 mg. Note that this number assumes an exact value of 4\% for the amount of the drug left over after a 12-hour period: A more reasonable interpretation of the level of accuracy would be to report the solution of 260.4 mg. This level of accuracy renders negligible the contributions of the drug taken longer than 24 hours ago.

c. **From the previous two parts of the problem we know that the greatest amount of the antibiotic in Susan's body is about 260.4167 mg and it will occur right after she has taken the last dose. However, we saw that already after the fourth dose she had 260.416 mg of the drug in her system, which is only insignificantly smaller. So we can say that starting on the second day of treatment, twice a day there will be about 260.416 mg of the antibiotic in her body. Over the course of the following 12 hours the amount of drug will decrease to about 4% of the maximum amount, which is 10.4166 mg. Then the cycle will repeat.**

HANDOUT #6

Smarter Balanced Assessment System: Connecting the Mathematics Claims to Classroom Instruction

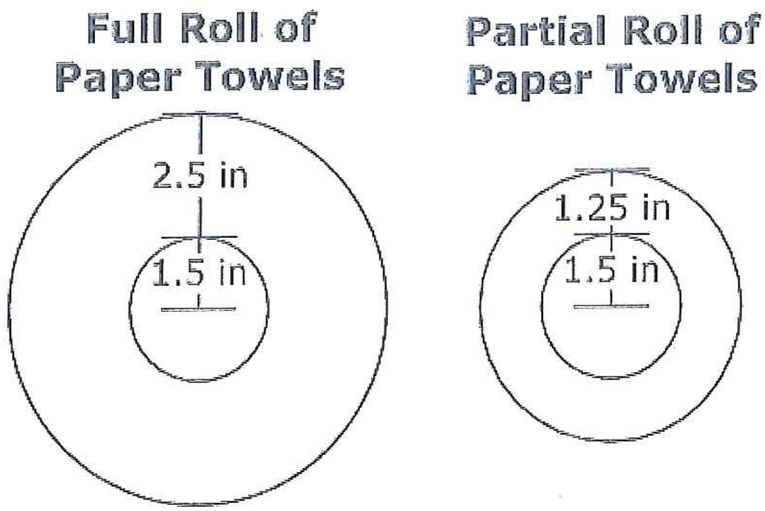
Bringing It Back to the Standards and the Teaching Practices

Standards for Mathematical Content Domain Cluster Heading Standard(s)	Standards for Mathematical Practice	Mathematics Teaching Practices	
		Teacher Actions	Student Actions

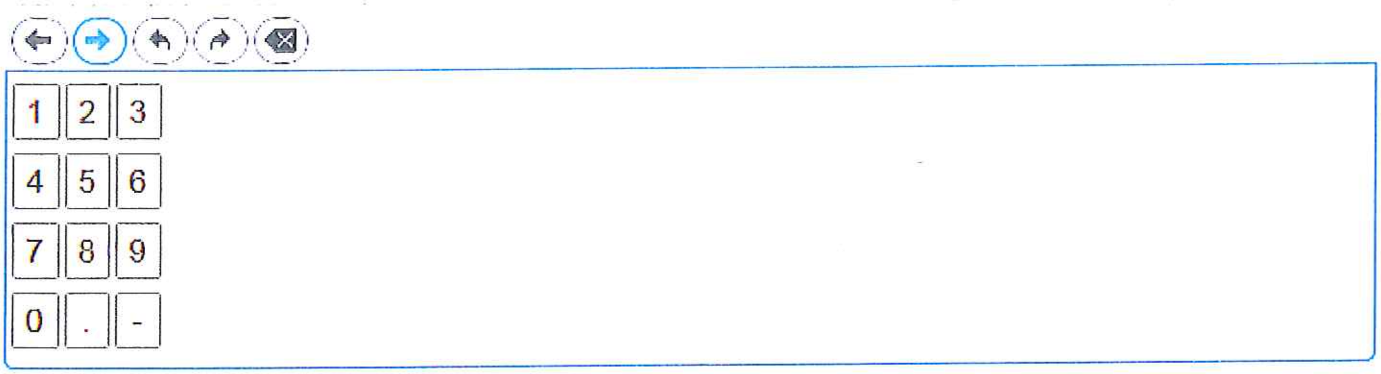
Item	Claim	Domain	Target	DOK	CONTENT	MP	Key
#22	4	G-MG	A	3	HSG.MG.A	2, 4	202-206

2032

The diagram shows the end view of a roll of paper towels when it is full and the end view of the roll after some of the paper towels have been used.



When the full roll of paper towels is unrolled, it has a length of 528 inches of paper towels of uniform width and thickness. Enter the length, in inches, of the paper towels remaining on the partial roll.



Key: 202 – 206, inclusive

Rubric: (1 point) Student enters a length within the given range.

RESOURCES

- CT Core Standards: http://ctcorestandards.org/
- Connecticut Dream Team 2014 Math Resources: http://ctcorestandards.org/?page_id=3771
- Achieve the Core: http://achievethecore.org/
- Illustrative Mathematics: https://www.illustrativemathematics.org/
- Smarter Balanced Assessment Consortium: http://www.smarterbalanced.org/
- Link to CSDE Student Assessment Smarter Balanced page: http://www.sde.ct.gov/sde/cwp/view.asp?a=2748&q=334488
- The Mathematics Assessment Project: http://map.mathshell.org/materials/index.php
- The Teaching Channel: https://www.teachingchannel.org
- Khan Academy: www.khanacademy.com
- The Balanced Assessment in Mathematics Program: http://balancedassessment.concord.org
- Illuminations by NCTM: http://illuminations.nctm.org/Default.aspx
- Inside Mathematics: http://www.insidemathematics.org/