

Meaningful mathematics experiences: What would your students say?

Ted Coe, Ph.D.





Daily attention to mathematical meanings

Programmatic attention to meaningful pathways

“The ants are my friends, they’re blowin’ in the wind .”

-not Bob Dylan

What does it mean to be *mathematically proficient*?

What does it mean to learn mathematics?

What would your students say?

Allow Me to Explain...

Mathematics learning, at a minimum, means learning:

- Ways of thinking
- Habits of thinking
- Ways of doing

Ways of Thinking (Example)

- How many years was it from December 7, 1941, to December 7, 2014?
- How did you figure it out?

Step 1: Notice the subtraction exercise.

$$2014 - 1941$$

Step 2: Set up the problem in a template.

$$\begin{array}{r} 2014 \\ -1941 \\ \hline \end{array}$$

Step 3: Work from right to left. Subtract 1 from 4.

$$\begin{array}{r} 2014 \\ -1941 \\ \hline 3 \end{array}$$

Step 4: Got it. Now move one column to the left. Subtract 4 from 1.

Step 5: Realize that I cannot subtract 4 from 1.

Step 6: Remember that I need to “borrow” from the next column to the left.

Step 7: Notice there is nothing to borrow. (Perhaps also consider how I wasn’t going to ever give it back anyway.)

Step 8: Borrow from the 2 in the far left column. Make it a 1.

Step 9: Think of the “0” in the top row as a 10. Wedge a tiny 1 between the 2 and the 0. Some might want to make it a little higher than the 0.

Step 10. Now borrow from the newly made 10. Make it a 9 by crossing out the 0 and 1 you just created during the previous step.

Step 11. Wedge a tiny 1 to the left of the original 1. I now have something that looks like this:

$$\begin{array}{r} 19 \\ \cancel{20}14 \\ - 1941 \\ \hline 3 \end{array}$$

Step 12. Subtract 4 from the newly-made 11. Get 7.

Step 13. Subtract 9 from 9.



Step 14. Subtract 1 from 1.

Step 15. Declare the answer to be 73.

What do we mean by *mathematical rigor*?

What would your students say?

Definition

- 1 a (1) : **harsh inflexibility** in opinion, temper, or judgment : **SEVERITY** (2) : the quality of being **unyielding** or inflexible : **STRICTNESS** (3) : severity of life : **AUSTERITY**
b : an act or instance of strictness, severity, or **cruelty**
- 2 : **a tremor caused by a chill** 
- 3 : a condition that **makes life difficult**, challenging, or uncomfortable; *especially* : extremity of cold
- 4 : strict precision : **EXACTNESS** • logical *rigor*
- 5 a *obsolete* : **RIGIDITY, STIFFNESS**
b : rigidness or torpor of organs or tissue that prevents response to stimuli
c : **RIGOR MORTIS** 

The Shifts of the Common Core



Focus strongly where the standards focus.



Coherence: Think across grades and link to major topics within grades.



Rigor: In major topics, pursue conceptual understanding, procedural skill and fluency, and application with equal intensity.

Ohio's High School Math Pathways

Rigor: Students use mathematical language to communicate effectively and describe their work with clarity and precision. Students demonstrate how, when and why their procedures work and why they are appropriate. Students can answer the question, "How do we know?"

Rigorous courses are...	vs.	Rigorous courses are not...
Defined by complexity, which is a measure of the thinking, action or knowledge that is needed to complete the task.		Characterized by difficulty, which is a measure of effort required to complete a task.
Measured in depth of understanding.		Measured by the amount of work.
Opportunities for precision in reasoning, language, definitions and notation that are sufficient to appropriate age/course.		Based on procedure alone.
Determined by students' processes.		Measured by assigning difficult problems.

What is *meaningful mathematics discourse*?

What would your students say?

NCTM Teaching Practices

Effective Mathematics Teaching Practices

Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.

Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.

Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Meaningful mathematics
discourse must center
on mathematical
meanings!

Fractions

What is a fraction?

Is a fraction one number or two numbers?

Angles

What is an angle?

What attribute are we measuring when we measure angles?

Division

- What different things does division do?
- How can you make sense of $12 \div 3$?
- How can you make sense of $12 \div \frac{1}{3}$?
- How can you make sense of $12 \div -\frac{1}{3}$?

You have three broomsticks

The RED broomstick is three feet long

The YELLOW broomstick is four feet long

The GREEN broomstick is six feet long



$$3 \neq 2$$

The RED broomstick is three feet long



The YELLOW broomstick is four feet long



The GREEN broomstick is six feet long



How much longer is the GREEN broomstick than the RED broomstick?

3 FEET

2 TIMES AS
LONG

The RED broomstick is three feet long



The YELLOW broomstick is four feet long



The GREEN broomstick is six feet long



1. How much longer is the GREEN broomstick than the RED broomstick?

3 FEET	2 TIMES AS LONG
--------	-----------------

2. How much longer is the YELLOW broomstick than the RED broomstick?

1 FOOT	?
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3. The GREEN broomstick is $\frac{6}{4}$ times as long as the YELLOW broomstick.



6 copies of $\frac{1}{4}$ of yellow
 $6 \cdot \frac{1}{4} = \frac{6}{4}$

The YELLOW broomstick is $\frac{4}{6}$ times as long as the GREEN broomstick.



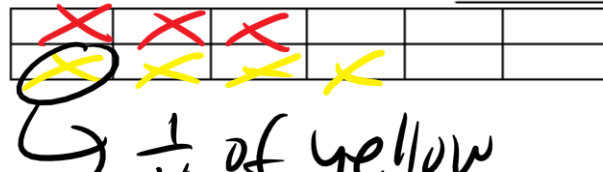
4 copies of $\frac{1}{6}$ of green
 $4 \cdot \frac{1}{6} = \frac{4}{6}$

The YELLOW broomstick is $\frac{4}{3}$ times as long as the RED broomstick.



$4 \cdot \frac{1}{3} = \frac{4}{3}$

The RED broomstick is $\frac{3}{4}$ times as long as the YELLOW broomstick.



$3 \cdot \frac{1}{4} = \frac{3}{4}$

A certain stock started at a value \$74. One year later it was valued at \$89.54. By what percent did the stock's value increase?

$$\frac{89.54}{74}$$

How many COPIES
OF 74 ARE IN
89.54? 1.21

Measurement

What do we mean when we talk about “measurement”?

Is area a measure?

...or is it something we can measure?

True or False:

- The only way to measure area is with another area

Measurement

“Technically, a *measurement* is a number that indicates a comparison between the attribute of an object being measured and the same attribute of a given unit of measure.”

—Van de Walle (2001)

But what “comparison”?

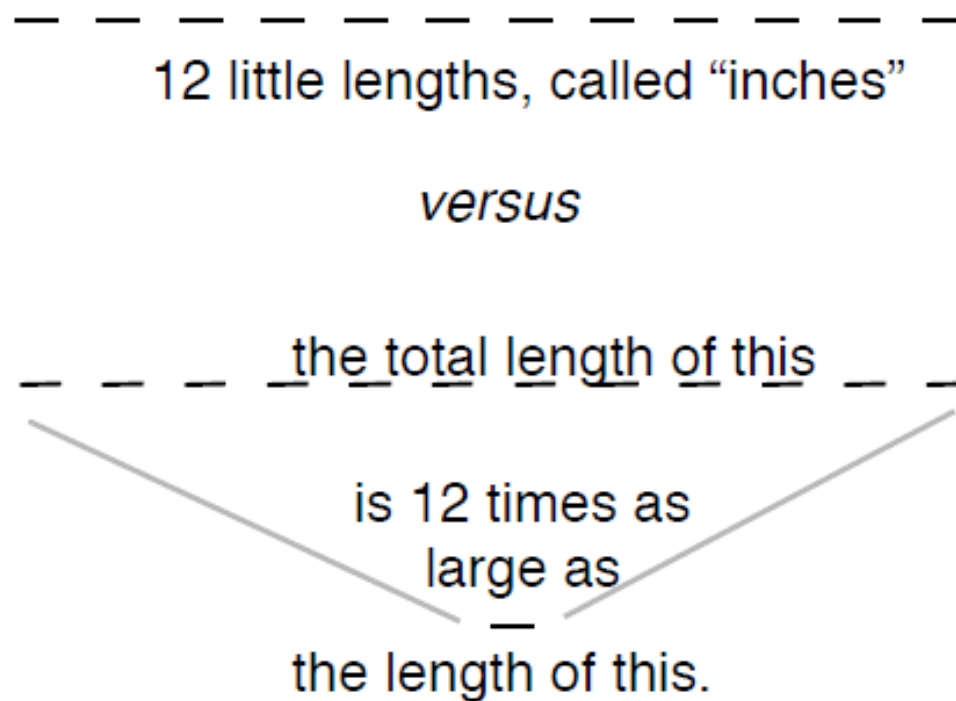


Figure 4. Measure as a number of things vs. measure as a ratio comparison.

Measurement

How about this?

- Determine the attribute you want to measure
- Find something else with the same attribute. Use it as the measuring unit.
- Compare the two: *multiplicatively*.

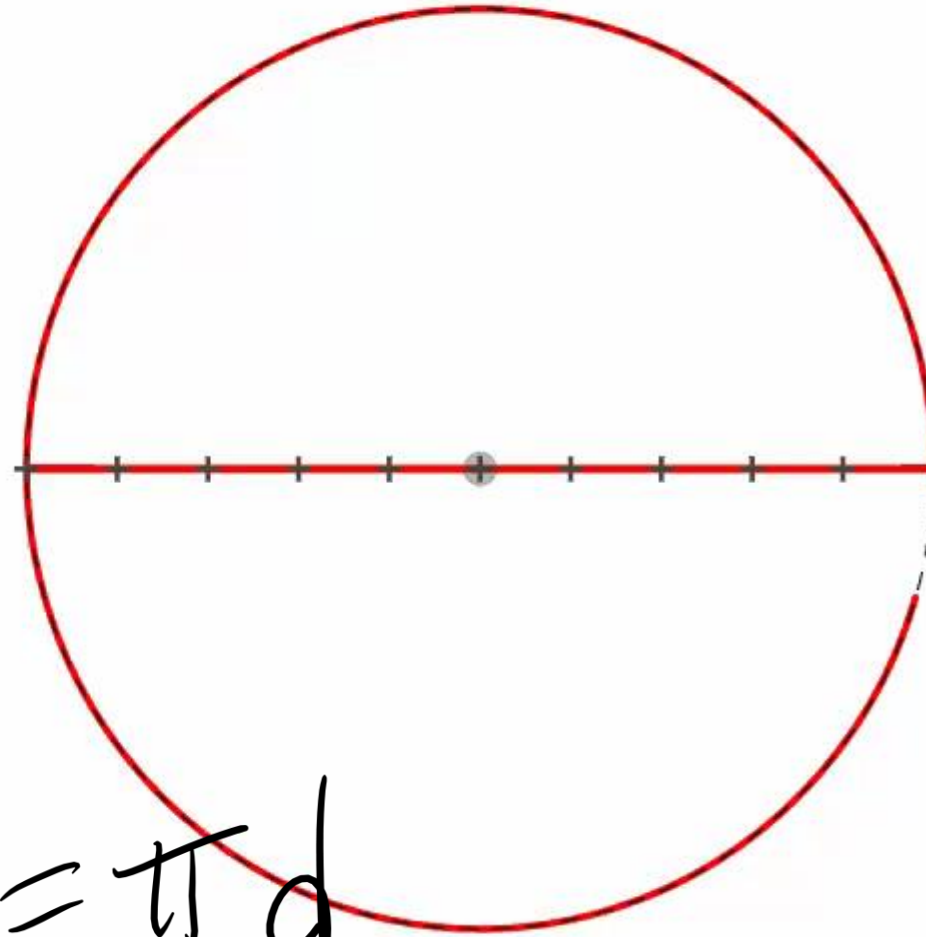
Adjust Radius ———●—

Pull Diameters ●———

Spin the bit ●———

$$\frac{C}{d} = \pi$$

$$C = \pi d$$

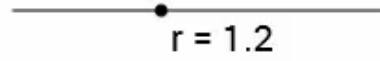


- The circumference is three and a bit times as large as the diameter.

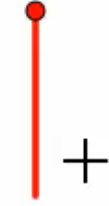
Adjust Arclength:



Adjust Radius:

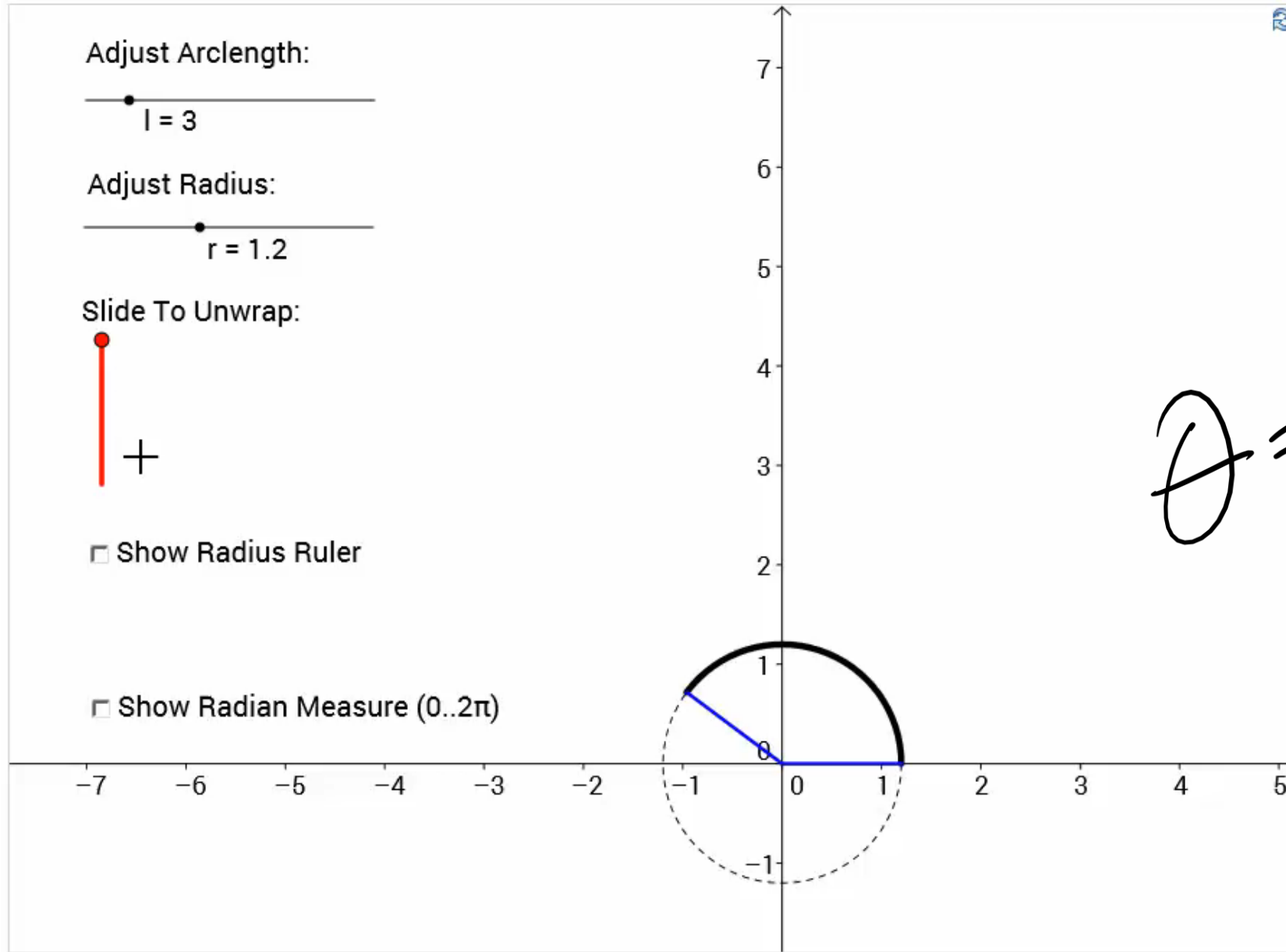


Slide To Unwrap:



☐ Show Radius Ruler

☐ Show Radian Measure ($0..2\pi$)



$$\theta = \frac{s}{r}$$

Rise over run

- Why divide?
- Related: What's the difference between a rate and a ratio?

Proportional Relationships

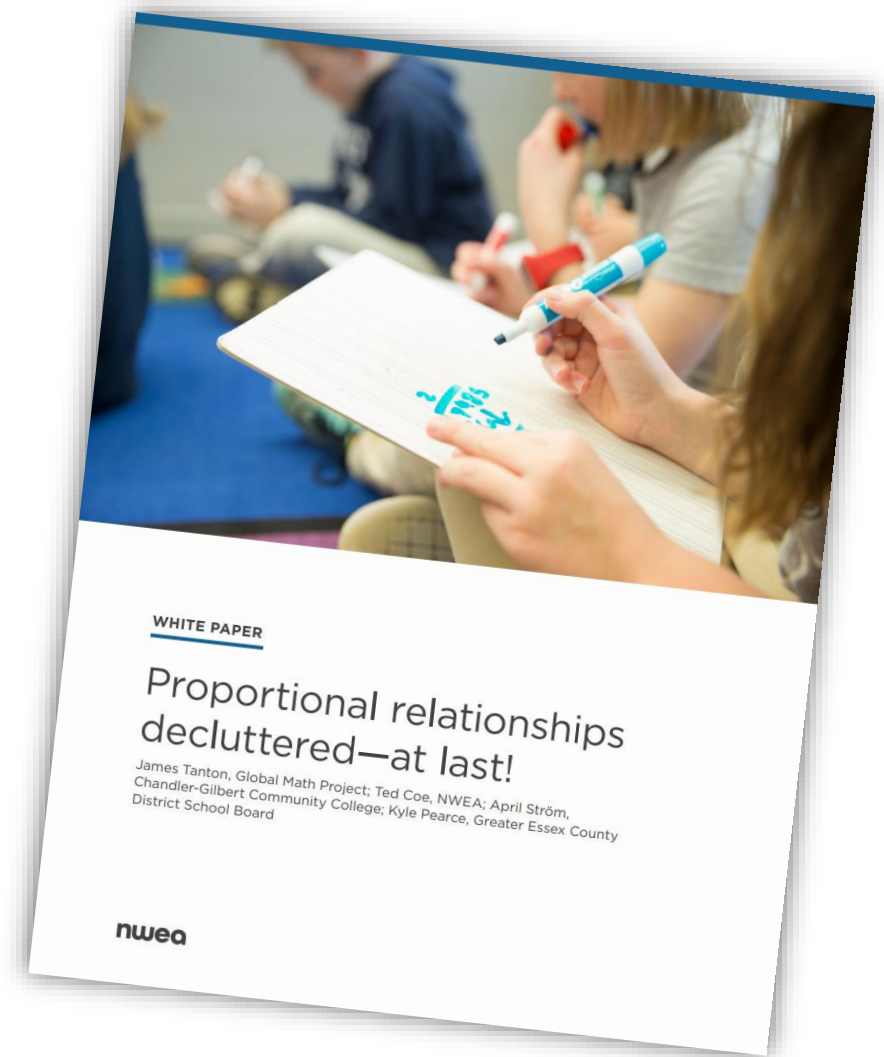
- What is a proportional relationship?
- What does it mean in general to say that one quantity is proportional to another quantity?*

“The confusing jumble of responses here is disturbing. At the very least it points to a lack of a common understanding within the school mathematics community of this very basic and important subject.”

Proportional Relationships Decluttered

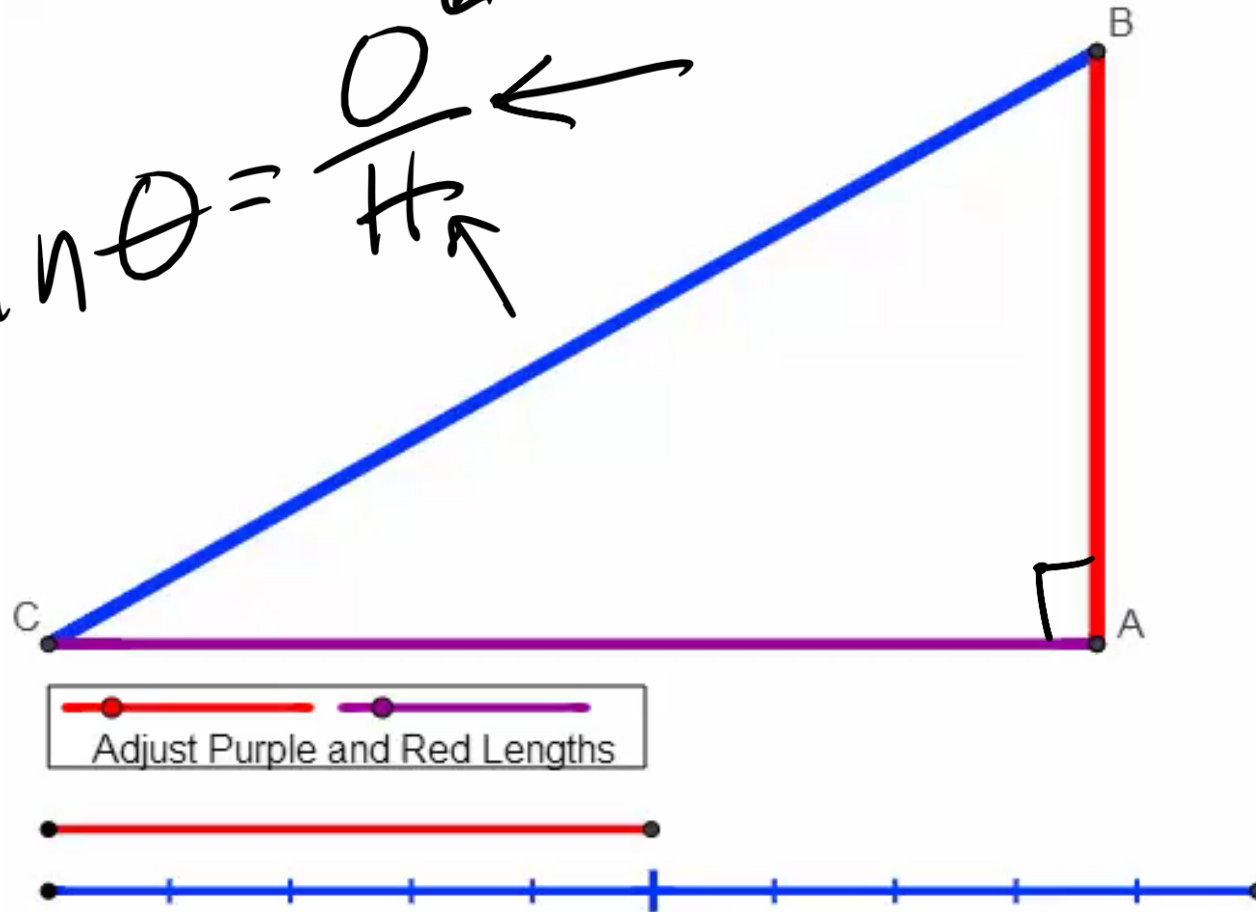
Many common scenarios involve two or more quantities that we can naturally measure,

- whose measures can or do adopt a variety of possible values,
- whose measures seem related to each other, and
- the measures “scale in tandem.”



$$\sin \theta = \frac{O}{H}$$

Handwritten formula for sine: $\sin \theta = \frac{O}{H}$. Arrows point from the 'O' to the opposite side (red) and from the 'H' to the hypotenuse (blue).



- Suppose you have a bank account that grows from \$60 to \$103.68 over three years.
 - If the account is growing linearly, what can you tell me? Stick to the quantities and avoid formulas.
 - If the account is growing exponentially, what can you tell me?

	Linear (additive)
Total Change:	

Pointless Points and Plotless Plots?

- Attend to meanings, even those that seem simple
 - What do you mean by...?
 - Never let an “it” go unchallenged
- Focus on ways of thinking



“All the lonely Starbucks lovers, they'll tell you I'm insane”
-*not* Taylor Swift

Daily attention to mathematical meanings

Programmatic attention to meaningful pathways

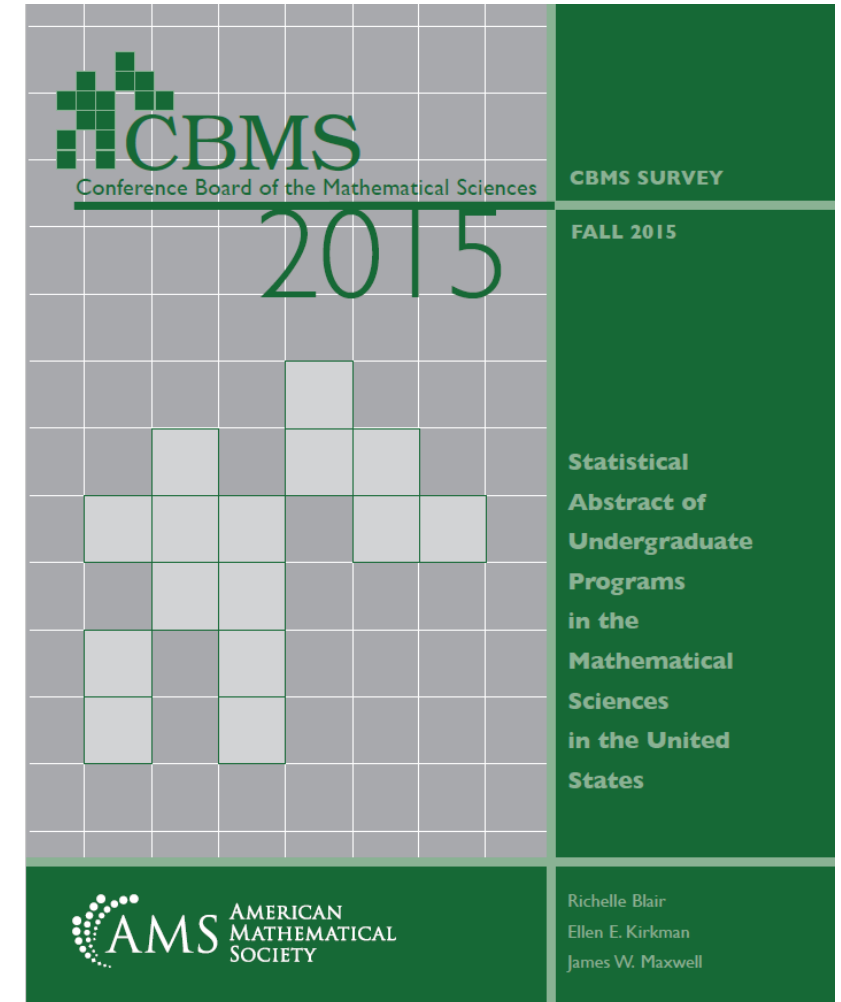
College Ready?

What did it mean to be “college ready” in mathematics in 2010?

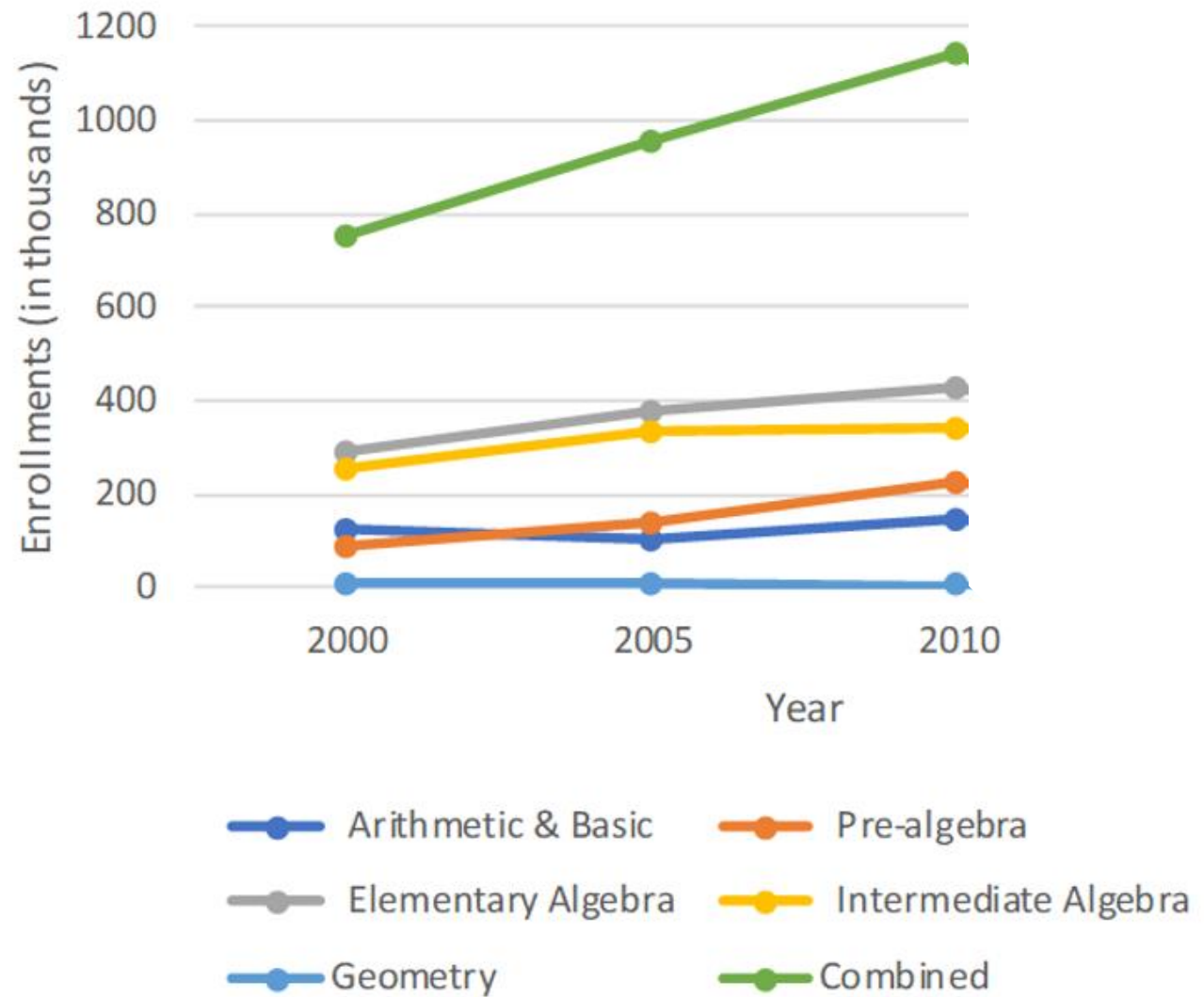
What will it mean to be “college ready” in mathematics in 2030?

CBMS 2015 Survey

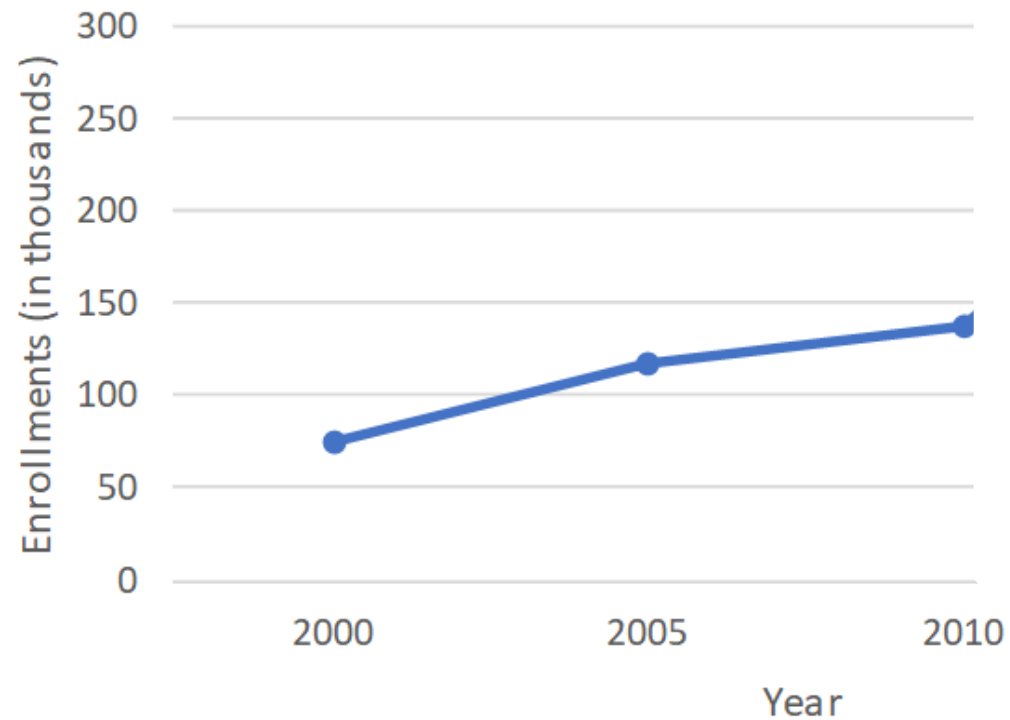
- In fall 2015, 58% of two-year colleges reported having implemented a pathways sequence



Two-year pre-college level enrollments



Two-year college Statistics and Probability enrollments.



University System of Georgia

Math Pathways



Who?	All majors other than those listed to the right	Majors that require calculus at some point in the sequence	Science, Technology, Mathematics majors	Engineering majors and all Georgia Tech students
Area A2 Math course>>	Quantitative Reasoning OR Mathematical Modeling	College Algebra	Pre-calculus OR Trigonometry	Calculus
Next Math course>>	Statistics	Pre-calculus>>> Calculus	Calculus	More Calculus

University System of Georgia

Start Smart – Choose the Right MATH Pathway!



Math Course	MATH 1001 Quantitative Reasoning OR MATH 1101 Introduction to Mathematical Modeling (OR MATH 1401 Elementary Statistics ¹)		MATH 1111 College Algebra	MATH 1112 College Trigonometry OR MATH 1113 Pre-calculus	Calculus
What majors start here?	Anthropology Business Administration ² Communication Criminal Justice ¹ Dental Hygiene Digital Media Educator Preparation Birth-to-Five Early Childhood Middle Grades (except Math and Science) P-12 Programs (except Health & Physical Education) Secondary (except Math and Science) English Film/Media Studies Foreign Languages General Studies ¹ Health Information Management ³	History ¹ International Studies ¹ Journalism/ Mass Communication ¹ Music Nursing Occupational Therapy Philosophy ¹ Political Science ¹ Psychology ¹ Public Relations ¹ Recreation Respiratory Therapy Social Work ¹ Sociology ¹ Speech Theatre Visual Arts	Business Administration ² Exercise and Health Science Exercise Science Geography (B.A.) Health and Physical Education (P-12) Health Information Management ³ Information Science Information Technology Logistics ⁴ Radiological Sciences	Agricultural Science ⁵ Architecture Astronomy Biology Chemistry Computer Science Educator Preparation Mathematics (Middle Grades or Secondary) Science (Middle Grades or Secondary) Engineering Technology Environmental Science Forestry Geography (B.S.) Geology Mathematics Pharmacy Physical Therapy Physics	Engineering ALL Georgia Tech students
¹ Students in these majors may take MATH 1001, MATH 1101, or MATH 1401. ² Business Administration recommendations are institution-specific. See the chart below for Business Administration recommendations for each institution. ³ Some Health Information Management programs require MATH 1111 College Algebra. Students should check institutional catalogs. ⁴ Some Logistics programs require MATH 1111 College Algebra. Students should check institutional catalogs.					

AMATYC (2014)

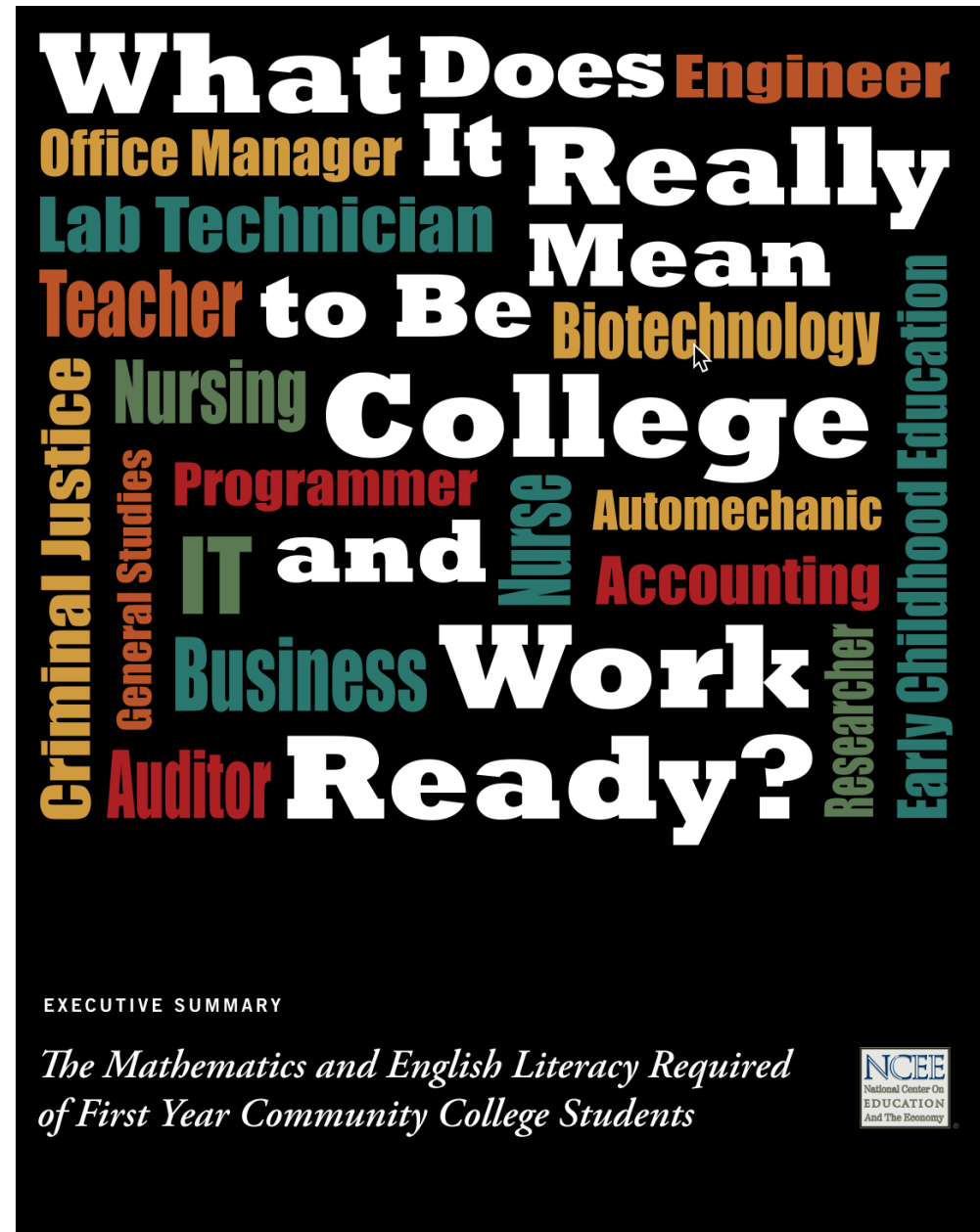
- The equivalent content in intermediate algebra courses is not required to master the content for most college-level mathematics courses that do not lead to calculus;

ACCUPLACER

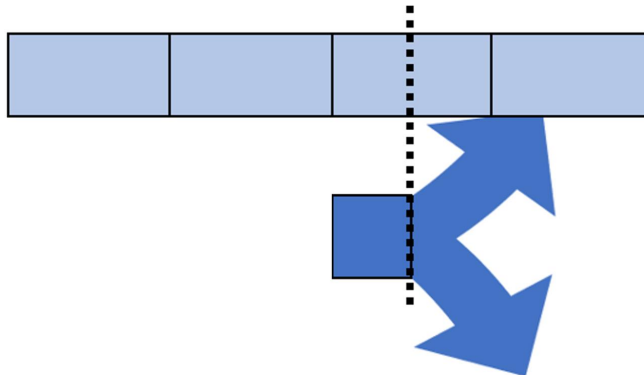
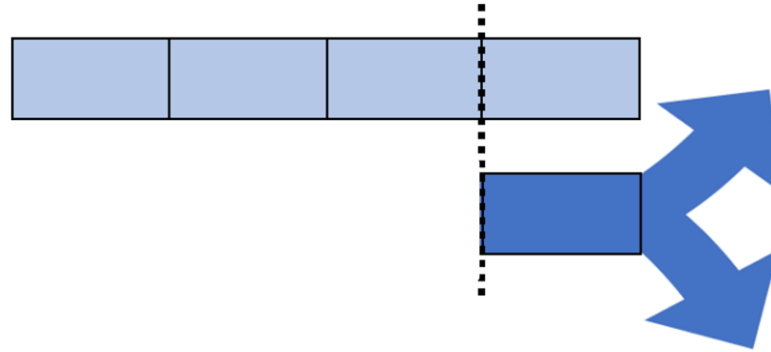
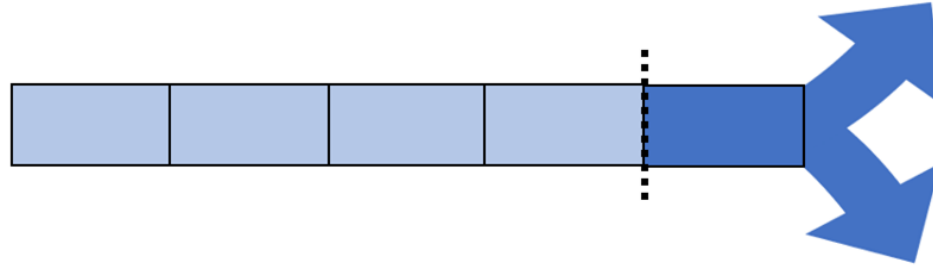
Next-Generation Quantitative Reasoning, Algebra, and Statistics (QAS) assesses the following knowledge and skills:

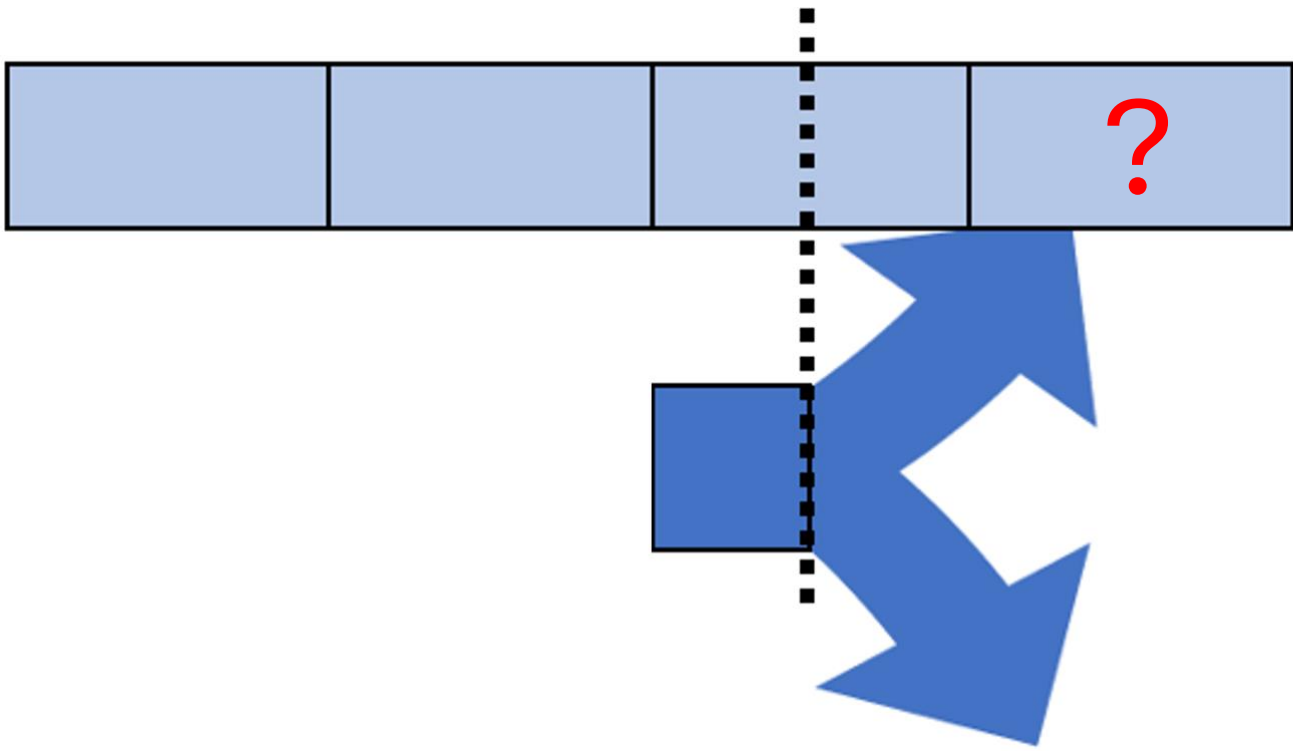
- Rational numbers
- Ratio and Proportional Relationships
- Exponents
- Algebraic Expressions
- Linear Equations
- Linear Applications and Graphs
- Probability Sets
- Descriptive Statistics
- Geometry Concepts

“...the mathematics that most enables students to be successful in college courses is not high school mathematics, but middle school mathematics, especially arithmetic, ratio, proportion, expressions and simple equations.”



An oversimplified, yet illustrative picture:





Accelerating to *where*?

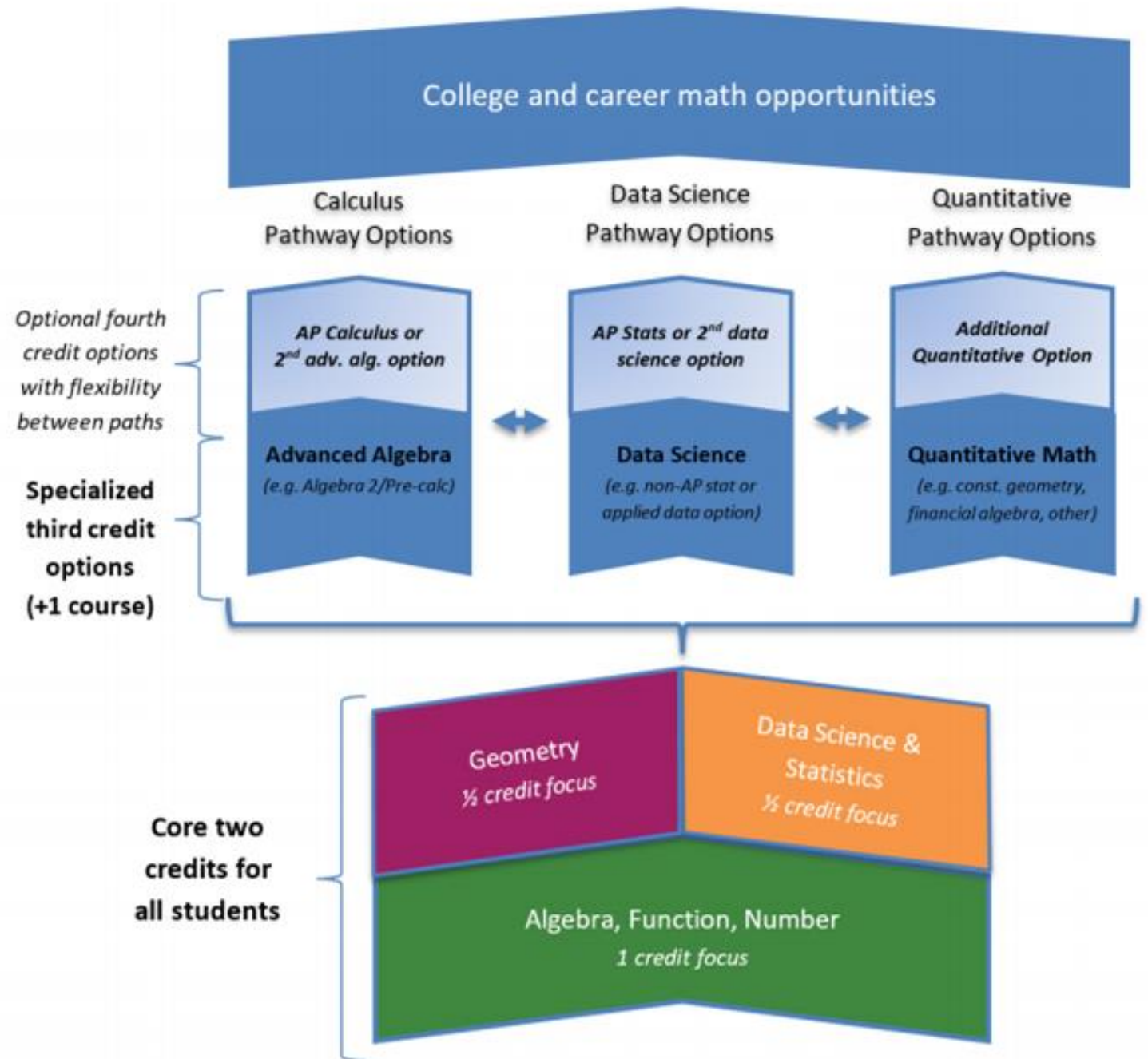
What would your students say?

MAA/NCTM, 2022: “A high school calculus course should not be the singular end goal of the PK–12 mathematics curriculum at the expense of providing a broad spectrum of mathematical preparation.”

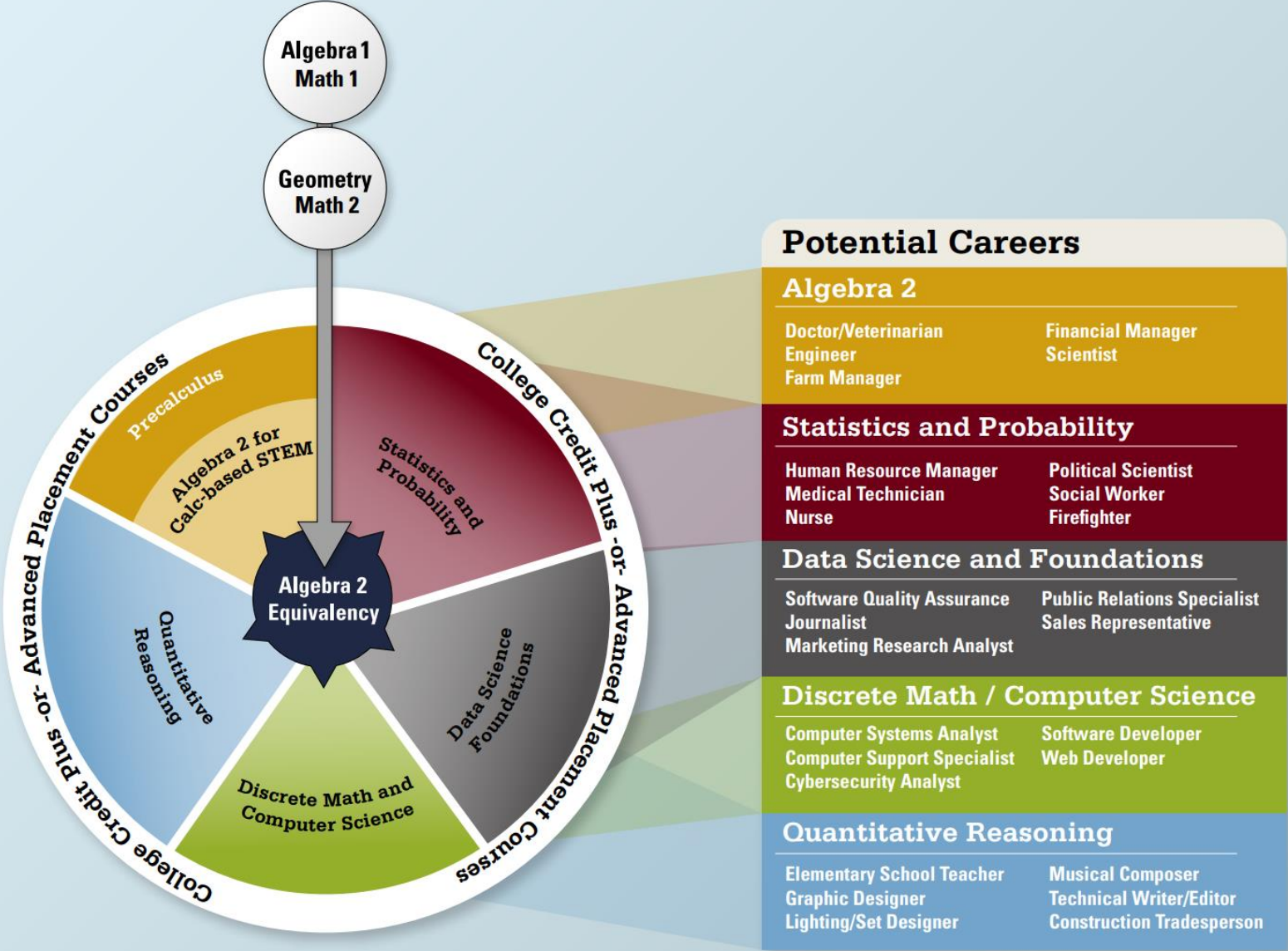
Accelerating to *where*?

“About a fifth of those who have studied calculus in high school earn and take advantage of advanced placement, starting in college with Calculus II or higher. About a third will retake Calculus I.”

Oregon 2+1 (2021)



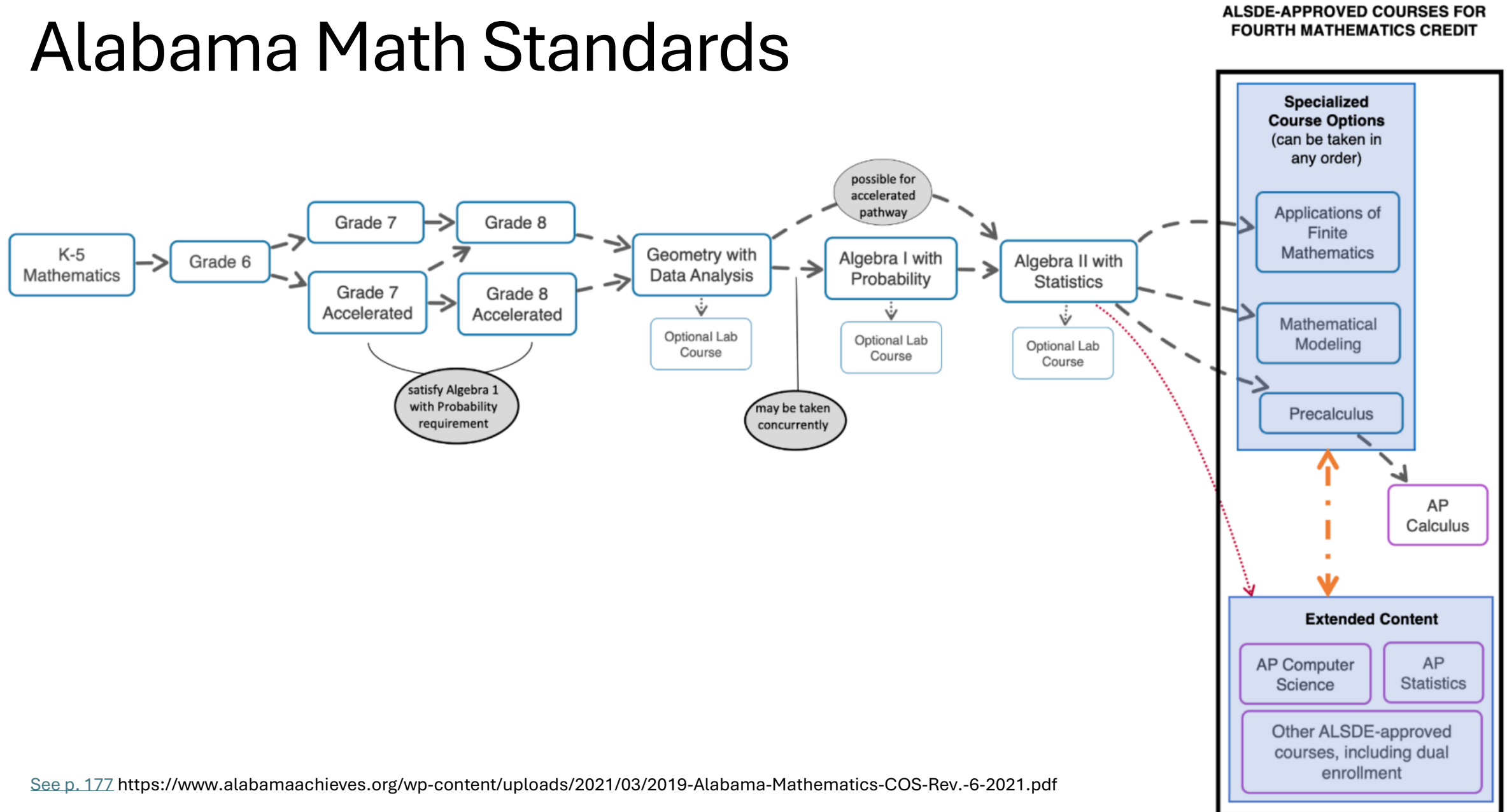
Ohio's High School Math Pathways



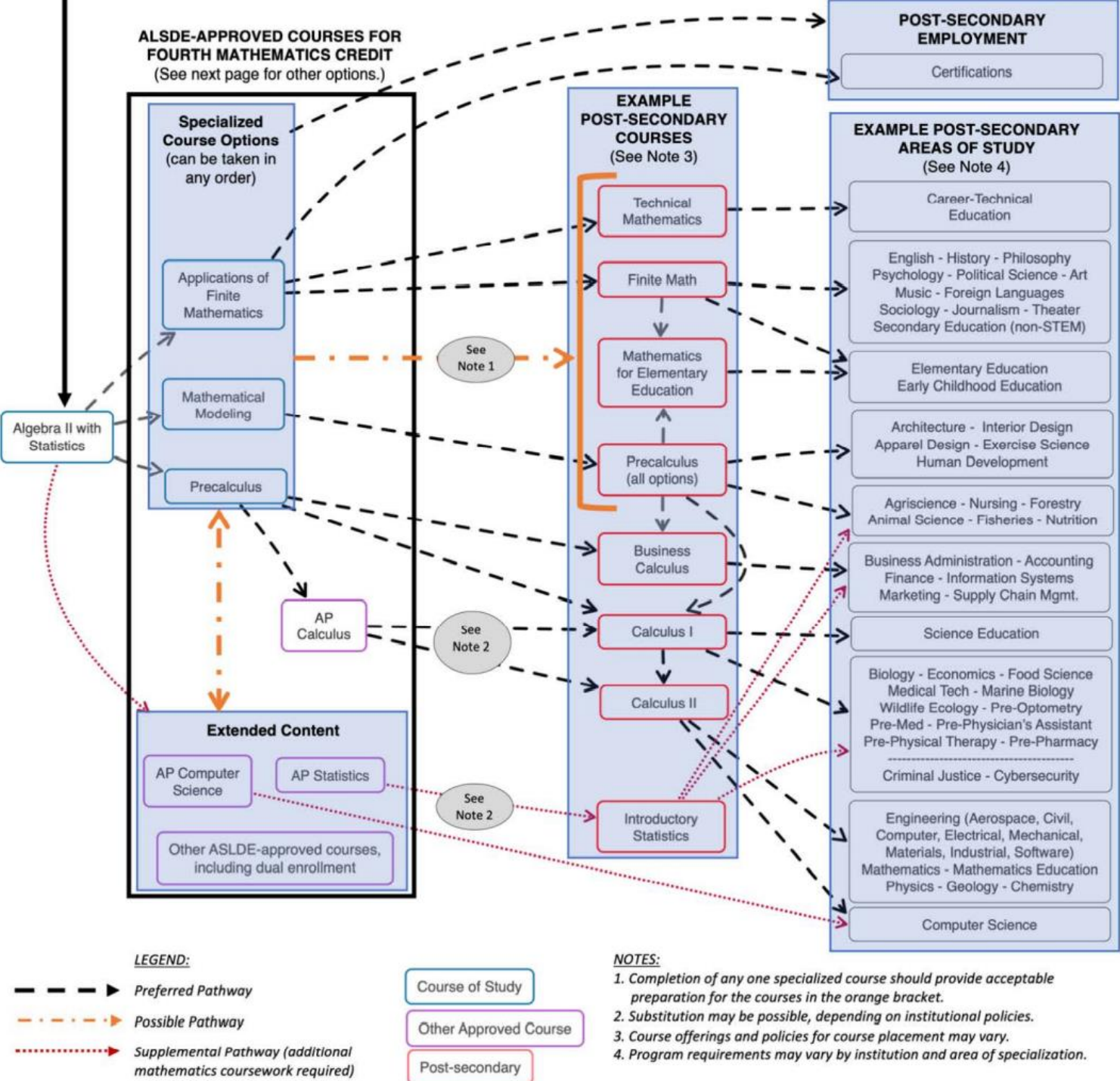
Districts may offer 1 or more courses listed in addition to Algebra 2.

Chart 1: Pathways through K-12 Mathematics

Alabama Math Standards



Alabama



Washington State: Modern Algebra II

Quantification

Equivalence

Variance

Algebraic
Functions

Mathematical
Modeling

Statistics,
Data Science

Quantitative
Reasoning

Advanced
Algebra

- Relevant, adaptable content
- Utilizes equitable teaching practices
- Meets university expectations for readiness and admission
- Supports Washington high school students on a variety of graduation pathways

IDAHO MATH TRANSITIONS

The Idaho Math Transitions project supports Idaho high schools offering math courses aligned with college and career interests, with the goal of advancing opportunities in mathematics for students. High school juniors and seniors may take math courses based on their chosen career pathway and maximize dual-credit general education math courses offered by Idaho's public colleges and universities.

72%
REQUIRE
OTHER
MATHEMATICS

28%
REQUIRE
CALCULUS

FRESHMAN ENROLLMENT
IN FOUR-YEAR MAJORS
(BURDMAN, 2015)

ARTS, TRADES,
COMMUNICATIONS,
LANGUAGES

HEALTH, BUSINESS,
SOCIAL SCIENCES

STEM FIELDS

DC
TECHNICAL
MATH

AP
STATISTICS

DC
ENGINEERING

DC/AP
COMPUTER
SCIENCE

DC/AP
CALCULUS

DC MATH IN
MODERN SOCIETY
MATH 123

DC STATISTICAL
REASONING
MATH 153

DC COLLEGE
ALGEBRA
MATH 143

DC/AP
PRE CALCULUS

NOT ALL COLLEGE
MAJORS REQUIRE
CALCULUS!

TWO YEAR
FOUR YEAR
CERTIFICATION
MILITARY ED OPPORTUNITIES
APPRENTICESHIP



Idaho State
Board of Education

IDAHO
Division of Career
Technical Education

GRADE 11

STUDENT CHOICES

GRADE 12

GRADE 9

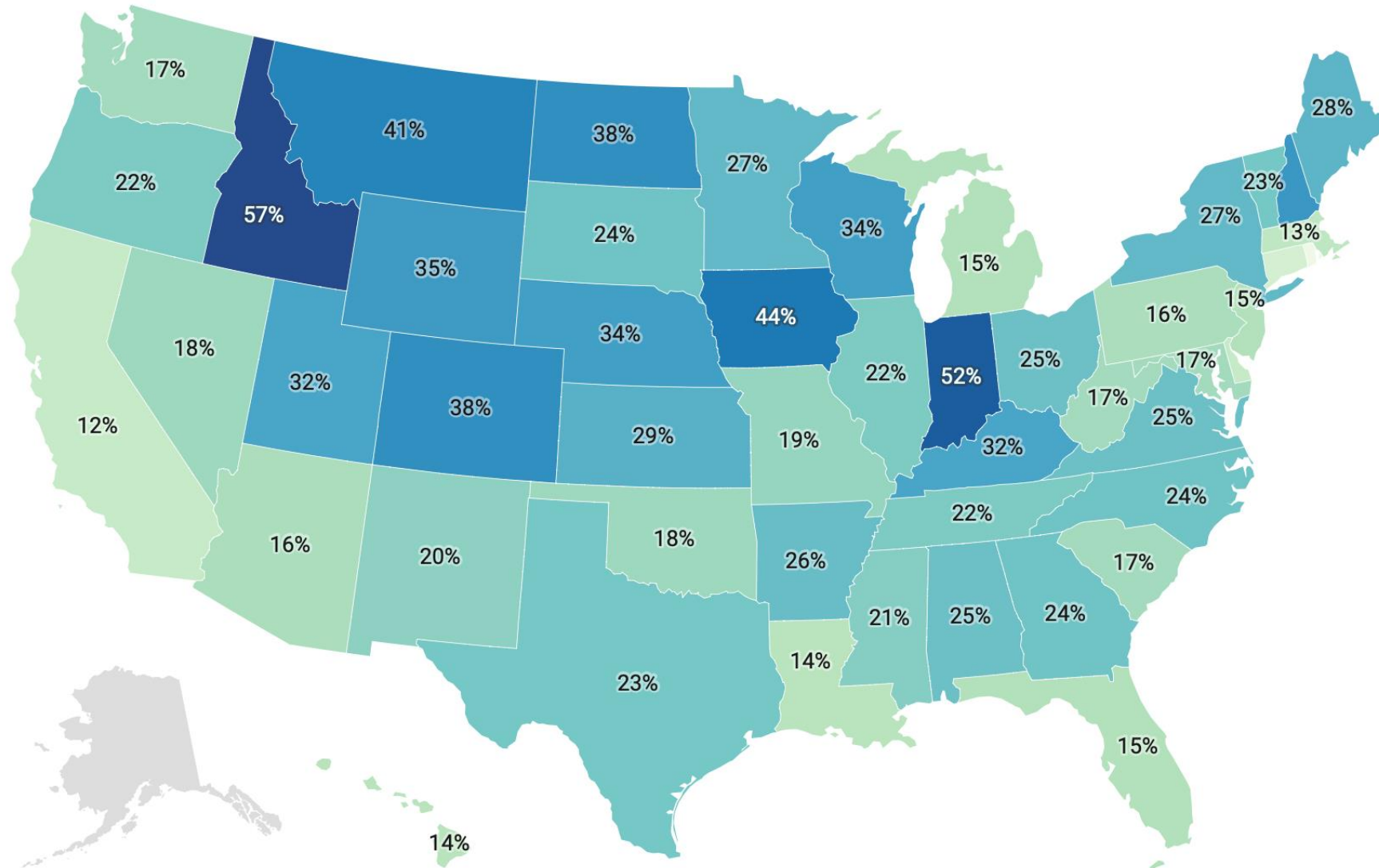
STANDARDS FOR ALL STUDENTS

GRADE 10

CONTENT INCLUDES ALGEBRA, GEOMETRY, STATISTICS
FOUNDATIONAL COURSES

Percentage of Community College Students in High School in 2022-23

Dual enrollment as a percent of
community college headcount



Get the data • Created with Datawrapper

<https://ccrc.tc.columbia.edu/easyblog/how-many-students-are-taking-dual-enrollment-courses-in-high-school-new-national-state-and-college-level-data.html>

Maryland (March 2025)

All schools must shift from the current Algebra I – Geometry – Algebra II (AGA) secondary mathematics course progression to an Integrated Algebra course progression that launches students into their chosen mathematics pathway.

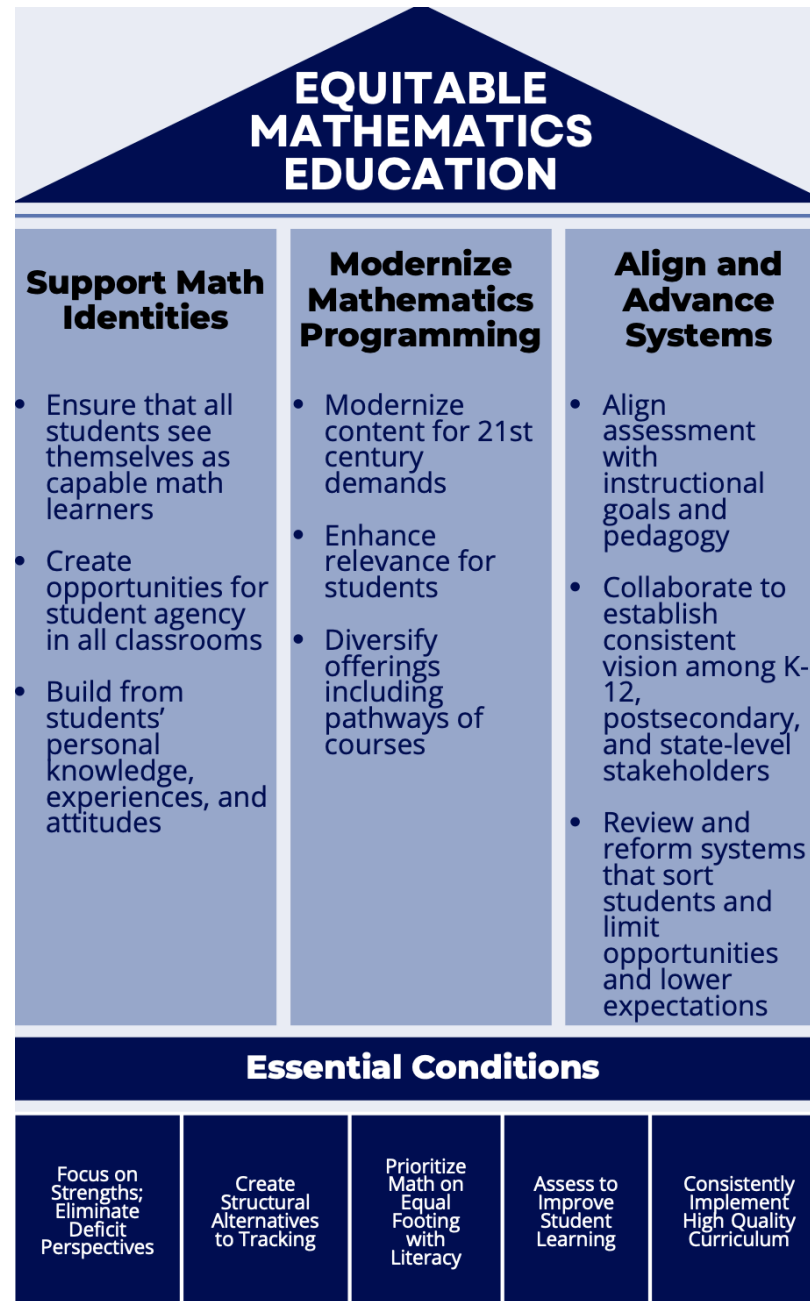
Beginning in SY 2027-2028, all courses beyond Integrated Algebra 1 and 2 must be aligned to secondary math pathway standards. Schools and districts must provide access to a variety of secondary mathematics pathways that align with different career and academic interests.

- From the March 2025 PreK-12 Mathematics Policy

Opportunities?

What would *you* say?

Connecticut Equity Statement (2023)



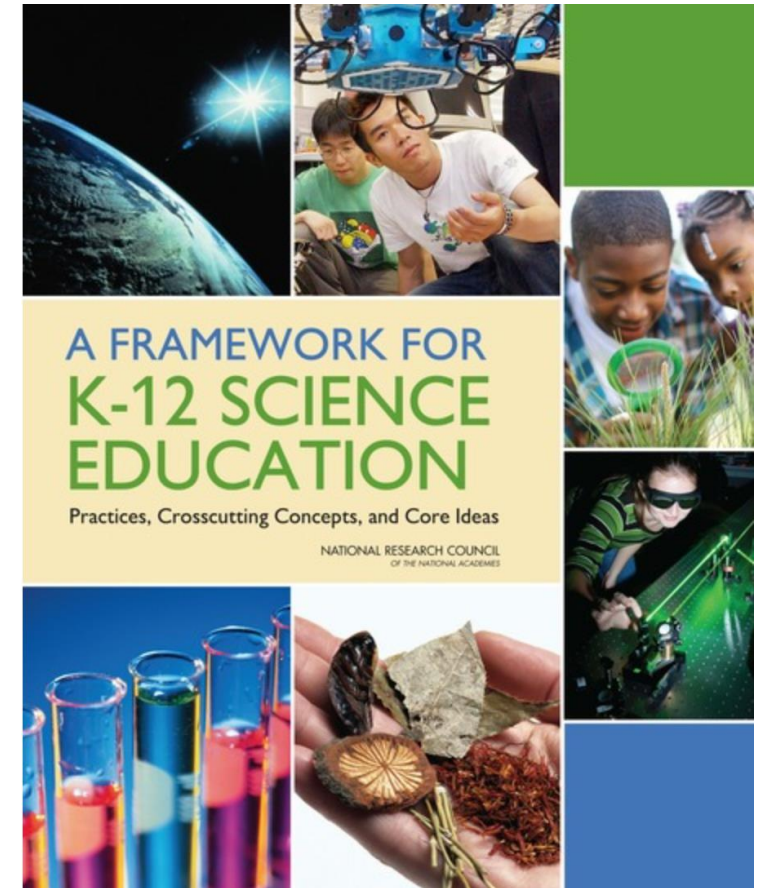
Essential Conditions

To achieve the goal of equitable mathematics education, we recognize that a foundation of essential conditions must be established.

1. Focus on Strengths - Eliminate Deficit Perspectives of Students
2. Create Structural Alternatives to Tracking
3. Prioritize Mathematics on Equal Footing with Literacy
4. Assess for Improving Student Learning
5. Consistently Implement High-Quality Curriculum

Depth over Breadth

“...an important role of science education is not to teach “all the facts” but rather to prepare students with sufficient core knowledge so that they can later acquire additional information on their own.”



What can we do?

- Ask about meanings, even those that seem simple
- Attend to ways of thinking
- Design meaningful pathways
 - Make sure all math courses go *somewhere*
 - Healthy messaging around acceleration

What would your students say?

- Math is a collection of disconnected facts and algorithms
 - **Math is coherent and founded on relationships of ideas**
- Math is to be memorized as given
 - **Math is about constructing knowledge**
- Math is idiosyncratic and situation specific
 - **Math is logical and systematic**
- Math is about quick answer-getting
 - **Math is about authentic problem solving**
- My effort has no impact on my ability to learn math: I am not a math person.
 - **My achievement depends on my persistence**

Thank you!

- ted@coequalmath.com
- tedcoe.com (classroom ready stuff)
- coequalmath.com



tinyurl.com/ctmath25

<https://www.tedcoe.com/math/ct2025/>