**Unit 1: Investigation 6 (3-4 Days)**

**Inverse Functions**

**Common Core State Standards**

F.BF.4 Find inverse functions.

F.BF.4a Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. For example, f(x) = 2(x3) for x > 0 or f(x) = (x + 1)/(x – 1) for x ≠ 1 (x not equal to 1).

F.BF.4b (+) Verify by composition that one function is the inverse of another.

F.BF.4c (+) Read values of an inverse function from a graph or a table, given that the graph has an inverse.

**Overview**

This Investigation introduces the concept of inverse functions. At the beginning of this investigation, inverse functions are considered in a less formal way, through the idea that “an inverse function undoes what a function does.” This leads students to the fact that the domain and range of a function are “reversed” for the inverse function. Students have considered 1-1 functions in Investigation 2 of Unit 1 (see Activity 1.2.3a and 1.2.3b). In this Investigation, they will see that only a 1-1 function will have an inverse relation that is a function. Students will find expressions to define inverse functions, restricting the domain of the original function when necessary. Finally, STEM-intending students will compose two functions to determine whether or not they are inverses of each other.

*Note: The written and internet literature are not consistent about the meaning of the word inverse when applied to functions. Some use inverse to mean inverse function, others to mean inverse relation which may or may not be a function. Once it is clear that students understand that only one-to-one functions can have an inverse function or if a function is not one-to-one but the domain is restricted so that on the restricted domain it is one-to-one you may want to shorten inverse function to just saying or writing inverse for this investigation. In this course inverse relations that are not functions are of no interest. We are interested in having inverse functions so using inverse to save writing inverse function all the time is a reasonable class agreement. Many well respected authors will make a statement such as a function R = f(x) has no inverse if it is not one-to-one for when that happens the value of x that cannot be uniquely determined from the value of R. We want students to recognize that we can always interchange the domain and range to create a new relation, but only when the relation is also a function can we use it to undo. So our focus is on inverse functions and therefor on 1 – 1 functions because they have inverse functions. The two functions f and f-1 convey the same information, but they express it differently. f-1(c)=d means f(d) = c.*

**Assessment Activities**

**Evidence of success: What will students be able to do?**

* Explain the meaning of the value of an output from both a function and an inverse function in context.
* Given a table, a graph, or a formula for a 1-1 function, be able to find the inverse function.
* Given a table, a graph, or a formula for a function that is not 1-1, be able to find an inverse function by finding an appropriate restriction of the domain of the function.
* Use functions and inverse functions to answer questions in realistic contexts.
* (For STEM intending students) Determine if two functions are inverses by composing the functions.

**Assessment Strategies: How will they show what they know?**

* **Exit Slip 1.6** asks students to determine if a function is 1-1. If so, students should be able to find its inverse function. If not, students will make a restriction of the domain and find an inverse function for the function for that restricted domain.
* **Journal Prompt 1** asks students to consider all of the functions in Algebra 2 so far and determine if any of them have inverse functions.
* **Activity 1.6.1 Functions and Their Inverses** provides students with a one-to-one function in context and asks them to give a verbal description of the value of an output of a function and the output of the inverse function This activity also asks students to provide a table of values of the inverse function given the function itself, and asks students to identify the graphs of a function and its inverse function.
* **Activity 1.6.2 Finding Inverses** asks students to find an inverse function for a given function after determining whether the function is 1-1 or not. If it is not, students find an appropriate restriction of the domain so that they can find the inverse function for that restricted domain.
* **Activity 1.6.3 Using Functions and their Inverses** provides several realistic situations in which students use both functions and inverse functions.
* **Activity 1.6.4 Are You My Inverse?** (+) asks students to find the composition of two functions to determine if they are inverse functions or not.

**Launch Notes**

Provide as many Rubik’s Cubes for the class as you are able to provide. Ask students to work on either scrambling them (if the cubes are in a solved state) or solving them (if the cubes are not solved). You might also show the following two-minute video showing some of the champion Speedcubers from 2014: <https://www.youtube.com/watch?v=QGt2GWofUCo> (as an alternative, you might show the following one-minute video from the rapper Logic, though it is a little harder to see the solution: <https://www.youtube.com/watch?v=ne7qPa_fSoQ>). After students have tried to solve the cube, point out that when a Speedcuber solves Rubik’s Cube, someone takes a cube in its solved state and scrambles the pieces. The Speedcuber then unscrambles the pieces to return the cube to its solved state. You might also take a solved cube and apply two “moves” to it (say, rotate the top level 90 degrees and then a side by 180 degrees). Show the cube to the class and ask them how to solve it. They should be able to determine that you would rotate the side by 180 degrees, then rotate the top level 90 degrees in the opposite direction from before. (You might point out that the solution not only makes the moves “backwards,” but also does them in the reverse order. This idea occurs in higher mathematics—for example, the inverse of the matrix AB, if it exists, is (AB)-1 = B-1 A-1.

One way to think of solving Rubik’s Cube mathematically is to think of applying a procedure to the solved cube to put it into a scrambled state. In order to get back to the solved state, one way would be to undo everything that was done to scramble the cube in the first place. We can think of this action as the “undo” procedure to return the cube to its solved state. If we have a function in mathematics, this “undo” function (if it exists) is called the inverse function of the original function. (Often shortened to inverse.)

**Teaching Strategies**

**Activity 1.6.1 Functions and Their Inverses** provides an introduction to the concept of inverse functions prior to students being asked to find an inverse function algebraically. The emphasis in this activity should be on the idea that “an inverse function undoes what a function does.” For example, if a function P(t) describes a population as a function of time, then P-1 is a function that describes time as a function of population. If P(2) = 10, then P-1(10) = 2. You should consider working through an example (with input from the class) where you create a table of values of a simple function (for example, f(x) = 0.3x2 for x ≥ 0) and plot the points by hand, then create a table of values of the inverse function and plot the function by hand on the same set of axes. Ask the class how the graph of the inverse function compares to the graph of the function itself. Students should see that the graphs are mirror images of each other across the line y = x. Point out that the set of inputs of the original function becomes the set of outputs for the inverse function, and the set of outputs from the original function becomes the set of inputs for the inverse function. This shows that the roles of the input (x) and the output (y) are “switched,” so that “x becomes y and y becomes x.” You might also plot the original function on a clear slide on top of an overhead projector, and then physically perform the motion that would change the input to output and output to input; this should confirm visually that the graph of the inverse function is a mirror image of the graph of the original function across the line y = x. We assume of course that you begin with a one-to-one function. You may want to discuss shortcuts for recognizing inverses functions. If you have a graph, students can hopefully note that is the graph is always increasing or always decreasing that the function will be one to one. You may also want to discuss the horizontal line test. But the major focus is having a 1-to-1 function and then determining its inverse by table, equation and graph and lastly using that inverse function to undo.

You should be sure to mention that given a one-to-one function f(x), the inverse function is usually written as f-1(x). It is important to point out that even though we write this with a negative exponent, f-1(x) does *not* mean 1/f(x).

**Differentiated Instruction (For learners needing more help)**

Students can have difficulty in creating a mirror image of a function like f(x) = -x + 4 (the third graph on question #3 of Activity 1.6.1) across the line y = x. Asking students to create a table of values to assist them, or inserting a blank table of values in this question, might help them visualize these graphs.

After students complete Activity 1.6.1, ask the class to answer the following question: Our next goal is to take a function y = f(x) and find a formula for the inverse function of that function. Based on the results of Activity 1.6.1, how can we do this?

**Pair Activity – Activity 1.6.2 – Finding Inverses**

Students work together to answer the question, “Can you always find an inverse relation for a function, or only sometimes? Will the inverse relation always be a function?”

While completing Activity 1.6.1, if you emphasized the fact that in an inverse relation the input and output “switch roles,” then students should be able to determine that an equation for an inverse function will be one where we “switch” the x and y in a formula . Students might have a harder time answering the question whether it is always possible to find an inverse function, and if the inverse relation is always itself a function. One hint you might give is if you used the example f(x) = 0.3x2 for x ≥ 0 in Activity 1.6.1, ask why you included x ≥ 0 in this function. If needed, ask the class what the graph of f(x) = 0.3x2 would look like if you take the mirror image of the graph across the line y = x. Students should see that this graph would not be the graph of a function. You can then point out the reason for discussing 1-1 functions back in Investigation 2; 1-1 functions have inverse functions but functions that are not 1-1 do not. In **Activity 1.6.2 Finding Inverses**, students sort a set of functions according to whether they are 1-1 or not. If they are, students find the inverse function; if they are not, students find a restriction of the domain of the beginning function so that they can find an inverse function. An inverse relation that is not a function is not helpful in undoing.

**Activity 1.6.3 Using Functions and their Inverses** includes a range of examples of realistic problems where inverse functions might be used. This activity could be done either in class or as a homework assignment. One question asks students to find a function of their own choice to convert a currency into dollars; they will need internet access to complete this question. Students can complete **Exit Slip 1.6** either before or after Activity 1.6.3.

Investigation 5 of Unit 1 is designed for STEM-intending students. **Activity 1.6.4 Are You My Inverse?** is a follow up to Investigation 5 and should be omitted if Investigation 5 was omitted. You might introduce this Activity by asking students this question: if we compose (“put together”) a function and its inverse, what should we get? Suppose f(a) = b. Or you might want to be even more concrete and start with f(2) = 3 for example. Suppose that we start with a value of a, then we evaluate f-1(f(a))? What will we get? Will we get the same thing if we evaluate f(f-1(b)). Then let f(x) = y and repeat. If a student understands that “an inverse function undoes what a function does,” they should see that both of these will lead back to the original input value they started with, so f(f-1(x)) = x, x from the domain of f-1 and f-1(f(x)) = x for x in the domain of f.

**Differentiated Instruction (Enrichment)**

Consider asking your students about the form of inverse functions as well. What kind of a function is the inverse of a linear function? A quadratic function once restricted? Can they provide a reason why?

Also consider giving students an exponential function. Can they find the graph of an inverse for the function f(x) = 2x? How about a formula? Of course, they will not be able to solve an equation like y = 2x unless they have seen logarithmic functions before! However, you might use this question to “foreshadow” what they will see in Unit 5.

Students can complete the Journal Prompt at the end of Investigation 6.

**Journal** **Prompt 1** Think about all of the functions you have seen and used in Algebra 2. Do all of these functions have an inverse relation that is also a function? Why or why not? Students may respond that linear functions do as long as the slope is not 0, absolute value functions do not, quadratic functions do not. They might add that the reciprocal function does.

Think again about all the functions you have seen and used in Algebra 2. Do all of these functions have an inverse that is also a function if we find an appropriate restriction of the domain of the function? Why or why not?

[Possible response: Students should state that not all functions have inverses where we assume the inverse is to be a function; a function needs to be 1-1 in order to have an inverse. They should also conclude that not all functions have inverses even if we restrict their domains; a constant function or a step function has no inverse unless we restrict the domain to a single point (for a constant function) or a small number of discrete points (for a step function).]

**Closure Notes**

Be sure to remind students about what inverses really are—“an inverse function undoes what a function does.” Ask students to define a function in context, determine if it is 1-1 and if it is, what its inverse is. Students should share their examples in class. Inverse functions (sometimes referred to as inverses) are of great importance in mathematics, and not just in functions—for example, they will see inverses when they learn about matrices in Unit 8. Students will meet inverses again in Unit 5 when they study the logarithmic family. If students can remember what inverse functions are conceptually and why we need a function to be one-to-one, they will be more likely to understand important ideas they confront in higher mathematics.

**Vocabulary**

Area of a circle

Celsius

Composition of functions

Cost of production

Currency conversion

Domain

Euro

Fahrenheit

Horizontal line test

Input and output of a function

Inverse function

Inverse relation

One-to-one (1-1) function

Radius of a circle

Range

Restricted domain

Tax table

**Resources and Materials**

**Activities 1.6.1, 1.6.2, and 1.6.3 should be completed in this Investigation by all students. In addition, Activity 1.6.4 should be completed by all STEM-intending students.**

Activity 1.6.1 Functions and their Inverses

Activity 1.6.2 Finding Inverses

Activity 1.6.3 Using Functions and their Inverses

Activity 1.6.4 Are You My Inverse?

Graphing calculator/computer software with a graphing utility for all activities

Graph paper for all activities

Online access for Activity 1.6.3

Rubik’s Cubes for all students

<https://www.youtube.com/watch?v=QGt2GWofUCo>

<https://www.youtube.com/watch?v=ne7qPa_fSoQ>)