**Unit 20 Circles and Spheres**

**(4-5 Weeks)**

***This unit incorporates material from Units 5 and 6 of the Connecticut Core Geometry Curriculum***

**UNIT PLAN**

In this unit a variety of methods, including formal synthetic proof, compass and straightedge constructions, and coordinate methods are applied to the study of circles and spheres.

In **Investigation 1 Radii and Chords (Geometry Unit 5 Investigation 2)** the concept of locus is applied to the perpendicular bisector of a line segment. Investigation 1 of Geometry Unit 5 has already been included in Unit 11. There the circle is defined as the locus of points that are at a given distance to a given points, from which the general equation $(x-h)^{2}+(y-k)^{2}=r^{2}$is derived.

We return to the locus definition of circle to show that the center of a circle lies on the perpendicular bisector of any chord. This allows us to locate the center of any circle given two chords and to circumscribe a circle about any triangle.

**Investigation 2 Central Angles and Arcs (Geometry Unit 5 Investigation 3)** examines central angles and arcs. Given the measure of a central angle in degrees, the ratio of the intercepted arc to the length of the radius is constant regardless of the size of the circle. This leads to the definition of radian as another way to measure angles. (This concept is introduced here and developed more fully in Algebra 2). The length of an arc is therefore given by $s=rθ$ where $θ$ is measured in radians. Similarly the area of a sector is given by $A=\frac{1}{2}θr^{2}$. Notice that when $θ=2π$ we have the circumference and area of the entire circle.

In **Investigation 3** **Tangents to Circles** (**Geometry Unit 5 Investigation 4)** we see an informal proof of the fact that a tangent to a circle is perpendicular to a radius drawn to the point of tangency. Two right triangles are congruent if their hypotenuses and one pair of legs are congruent, a theorem that will be used to prove that tangent segments drawn from an external point to a circle are congruent. One activity will revisit the equation of a circle in the coordinate plane and use the slopes of perpendicular lines to find the equation of a line tangent to a circle at a given point.

**Investigation 4 Angle Bisectors and Inscribed Circles** (**Geometry Unit 5 Investigation 5)** leads to the construction of the center of the inscribed circle in a triangle, using the property that the bisector of an angle is the locus of points equidistant from the sides.

In **Investigation 5 Inscribed Angles and Intercepted Arcs** (**Geometry Unit 5 Investigation 6)** we show that an inscribed angle in a circle is equal in measure to ½ the measure of its intercepted arc. A special case is Thales’ Theorem that the angle inscribed in a semicircle is a right angle. We will also show that the opposite angles of a cyclic quadrilateral are supplementary. An optional activity (addressing standard C-4+) will have students construct tangents to a circle from an external point.

We now turn our attention to three dimensions and study spheres, the solid analog of a circle.

**Investigation 6 Spheres (Geometry Unit 6 Investigation 5)** extends the informal study of the sphere found in Unit 5 to justify formulas for the surface area and volume of a sphere. Using Cavalieri’s principle it anticipates ideas that some students will explore more deeply in a calculus course and provides some beginning experience with applications of the formulas to real world problems.

**Investigation 7 Geometry on the Sphere (Geometry Unit 6 Investigation 6)** introduces students to a geometry different from the Euclidean model used so far. They learn that the shortest path through two points lies on a great circle and that the sum of the interior angles of a spherical triangles is greater than 180° but less than 540°. Students work with coordinate systems on a sphere which are related to latitude and longitude on the Earth’s surface. They are introduced to applications of spherical geometry to cartography and Global Positioning Systems (GPS).

**Essential Questions**

* How are the equations of conic sections related to their locus definitions?
* What relations among angles, chords, and tangents to circles can be proved?
* How are the lengths of arcs, areas of sectors, and radian measure related to central angles in circles?
* What are the properties of inscribed and circumscribed triangles and inscribed quadrilaterals?
* Where do the formulas for volume and surface area of a sphere come from?
* Is there only one kind of geometry, the Euclidean Geometry with which we are familiar?

**Enduring Understandings**

Most properties of circles may be derived from the definition of a circle as the locus of points at a given distance from a given point.

**Unit Contents**

Investigation 1 Radii and Chords (3 days)

Investigation 2 Central Angles and Arcs (2 days)

Investigation 3 Tangents to Circles (3 days)

Investigation 4 Angle Bisectors (2 days)

Mid-unit test (1 day)

Investigation 5 Inscribed Angles and Intercepted Arcs (2 days)

Investigation 7 Spheres (2 days)

Investigation 7 Geometry on the Sphere (optional, 2 days)

Performance Task (1-2 days)

End of Unit Test (1 day review, 1 day test)

.

**Common Core Standards**

*Mathematical Practices #1 and #3* *describe a classroom environment that encourages thinking mathematically and are critical for quality teaching and learning. Practices in bold are to be emphasized in the unit.*

1. Make sense of problems and persevere in solving them.

2. Reason abstractly and quantitatively.

3. **Construct viable arguments and critique the reasoning of others**.

4. Model with mathematics.

5. **Use appropriate tools strategically.**

6. Attend to precision.

7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning.

**Standards Overview**

* Understand and apply theorems about circles
* Find arc lengths and areas of sectors of circles
* Explain volume formulas and use them to solve problems.

**Standards**

G-CO.9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment’s endpoints.*

G-C.2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

G-C.3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

G-C 4. (+) Construct a tangent line from a point outside a given circle to the circle.

G-C.5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

G-GMD.2. (+) Give an informal argument using Cavalieri’s principle for the formulas for the volume of a sphere and other solid figures.

G-GMD.3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.★

G-GPE.1. Derive the equation of a circle of given center and radius using the Pythagorean

**Vocabulary**

arc

arc length

center

central angle

chord

circle

circumscribed circle (of a triangle)

cyclic quadrilateral

diameter

Equator

Equidistant

geodesic

great circle

hemisphere

inscribed angle (of a circle)

inscribed quadrilateral (of a circle)

inscribed circle (of a triangle)

intercepted arc

latitude

locus of points

longitude

meridian

major arc

minor arc

point of tangency

Prime Meridian

radian measure

radius

secant

sector

semicircle

sphere

spherical coordinates

spherical excess

spherical triangle

tangent

tangent segment

trilateration

**Theorems**

**Perpendicular Bisector Theorem:** the locus of points that are equidistant from the endpoints of a segment is the perpendicular bisector of the segment.

**Perpendicular Bisector of a Chord Theorem**: The perpendicular bisector of any chord of a circle passes through the center.

**Radius Chord Midpoint Theorem:** If a radius of a circle bisects a chord (that is not a diameter), then it is perpendicular to the chord.

**Radius Chord Midpoint Converse:** If a radius of a circle is perpendicular to a chord then it bisects the chord.

**Construction:** To circumscribe a circle about a triangle.

**Arc Length Theorem:** The length of an arc of a circle is equal to the product of the radius and the radian measure of the central angle that intercepts the arc.

**Area of a Sector Theorem:** The area the sector of a circle is equal to the half product of the square of the radius and the radian measure of its central angle.

**Shortest Distance Theorem:** The shortest segment joining a point to a line is the perpendicular segment.

**Radius-Tangent Theorem:** A tangent to a circle is perpendicular to the radius drawn to the point of tangency.

**Radius-Tangent Converse:** If a line is perpendicular to the radius of a circle at a point on the circle, then it is tangent to the circle.

**HL Congruence Theorem:** If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of another right triangle, then the triangles are congruent.

**Tangent Segments Theorem**: If two tangents are drawn to a circle from the same point outside the circle, the segments from that point to the circle are congruent.

**Angle Bisector Theorem:** The locus of points equidistant from the sides of an angle is the bisector of the angle.

**Inscribed Angle Theorem**: The measure of an angle inscribed in a circle is one-half the measure of the central angle that intercepts the same arc.

**Construction:** To inscribe a circle in a triangle.

**Thales’ Theorem:** An Angle inscribed in a semi-circle is a right angle

**Cyclic Quadrilateral Theorem:** The opposite angles of a cyclic quadrilateral are supplementary.

**Construction:** To construct the tangent segments to a circle from a point outside the circle (+)

**Hemisphere-Cylinder Theorem:** The volume of a hemisphere is equal to $\frac{2}{3}$ the volume of a cylinder with the same radius and height equal to its radius.

**Assessment Strategies**

**Performance Task**

Students will apply the properties of circles to determine the most efficient way to pack cylindrical cans, given the diameter of the can and the dimensions of the box.

You will need rolls of pennies for the introduction to the performance task. Give students page 1 only to begin. After they have discovered square and hexagonal packing, you may give them the remaining page. Note that there is a group component to this project.

**Other Evidence (Formative and Summative Assessments)**

* Mid-unit test
* End-of-unit test
* Exit slips
* Class work
* Homework assignments
* Math journals