**Unit 6: Investigation 2 (3 Days)**

**Unit Circle Definition of Trigonometric Functions**

**Common Core State Standards**

F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F.TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for π/3, π/4 and π/6, and use the unit circle to express the values of sine, cosine, and tangent for π - x, for π + x and 2π - x in terms of their values for x, where x is any real number.

**Overview**

Students will understand the relationship on the unit circle called the Wrapping Function that maps a real number t to the ordered pair at the terminal point determined by the arc of length t. Symbolically, W(t) = (x,y) . This point W(t) is also the point where terminal ray of the angle of measure t in standard position intersects the unit circle. To be precise in our language, we will want to use the phrase “directed distance” to indicate that distances to the left of the y axis or below the x axis will be negative, otherwise, directed distance is positive. Analogously, “directed measure of the leg of a right triangle” will be negative if the horizontal leg is on the negative x axis or if the vertical leg is below the x axis. The length of the hypotenuse that is the length of the radius of the circle is positive. Note that the right triangle is defined by the origin, the point (a,b) and the x axis.



 The students will discover that the following are equivalent:

* the cosine function of a real number t; the x coordinate of W(t); the directed horizontal distance of W(t) from the y axis; the ratio of ‘a’ the directed length of the leg adjacent to angle t to the length of the hypotenuse ‘1’.
* the sine function of a real number t; the y coordinate of W(t); the directed vertical distance of W(t) from the x axis; the ratio of ‘b’ the directed length of the leg opposite t to the length of the hypotenuse ‘1’.
* the tangent function of a real number t; the ratio of the y to the x coordinates of W(t), the slope of the terminal ray of angle t; the ratio ‘b/a’ that is the ratio of the directed length of the leg opposite t to the directed length of the leg adjacent to t .

Using the odd and even symmetry of the unit circle, and given the value of a trigonometric function of t, students will be able to find the trigonometric function of t added to any integer multiple of π.

**Assessment Activities:**

**Evidence of Success: What Will Students Be Able to Do?**

* Identify the point W(t) on the unit circle that is determined by arc measure ‘t’.
* Identify the point W(t) on the unit circle determined by the point where the terminal ray of the central angle t intersects the unit circle.
* Sketch the reference angle and corresponding triangle for any of the special triangles, and label the lengths of the sides of the triangle. From the triangle, use right triangle definitions of sine, cosine and tangent to find the trigonometric functions of any angle measured in degrees or radians (real number).
* From a single x or y coordinate of W(t) on the unit circle, identify the three trigonometric functions of t and the trigonometric functions of t plus any integer multiple of π.
* Identify the quadrantal angles, W(t) for the points where the unit circle intersects the x and y axes, and evaluate the trigonometric functions of the quadrantal angles. A quadrantal angle is one whose terminal ray lies on the x or y axis.
* Interpret the sine and cosine of t as the vertical and horizontal directed distances, of W(t) from the x and y axes, respectively. Identify the tangent of t as the slope of the terminal ray of angle t in standard position.

**Assessment Strategies: How Will They Show What They Know?**

* **Exit Slip 6.2.1** asks students for the domain, range and rule for the wrapping function. Then it has them draw a mapping diagram showing three sample inputs and outputs of the mapping function.
* **Exit Slip 6.2.2** asks students to tell one thing they learned today, and one thing they still don’t understand.
* **Exit Slip** **6.2.3** hasstudents evaluate sin t, cos t and tan t, given the x or y coordinate of a point on the unit circle in quadrant II for an unknown ‘t.’
* **Journal Prompt 1** asks students toexplain how the right triangle definitions for trigonometric ratios are related to the circular definitions and to explain how they are different. Students are asked to illustrate their explanations with the example: On a unit circle, sketch the angle 4π/3 in standard position on the coordinate plane, indicate the reference triangle, sketch the reference triangle, label the lengths of the sides of the reference triangle and find the 3 trigonometric functions of t.
* **Activity Sheet 6.2.1 Wrapping Function** has two parts. Part 1 contains some short exercises and drill problems to check that the students can use the vocabulary and the notation of the wrapping function, and can find a point on the unit circle when the (flexible) ruler used to wrap around the unit circle is scaled in terms of π. Part 2 presents students with a unit circle graphed on an x and y axis marked off in tenths of a unit and the circumference marked in tenths of a unit. The students will use the wrapping function to find W(t) when the unit of measure is in decimals.
* **Activity Sheet 6.2.2 The Circular Definition of Trigonometric Functions** is a group activity designed to assist students in formulating an approach to locating points on the unit circle and seeing that one way to approximate the trigonometric function values is the read off the coordinates of W(t).
* **Activity Sheet 6.2.3 Working with the Circular Definition of Trigonometric Functions** asks students to work with their new definitions of the trigonometric functions of any real number by presenting a variety of questions about circular functions and the unit circle.

**Launch Notes**

**The Wrapping Function**

Consider using a video lesson of the wrapping function in order to Launch this investigation. Have the students watch the 7 minute video by Susan Cantey on the website: Learn Zillion. The video is called “Understand the wrapping function using the unit circle”. Here is the direct link: <https://learnzillion.com/lessons/2771-understand-the-wrapping-function-using-the-unit-circle> . You can then replay the video with starts and stops at the places you want to have a class discussion and have students take notes.

Do one or two problems from Part 1 of **Activity Sheet 6.2.1 Wrapping Function** in class. Then do a few problems from Part 2 of **Activity Sheet 6.2.1.** You can assign the remainder as group work or for homework. Part 1 contains some short exercises and drill problems to check that the students can use the vocabulary and the notation of the wrapping function, and can find a point on the unit circle when the (flexible) ruler used to wrap around the unit circle is in terms of π. Part 2 presents students with a unit circle graphed on an x and y axis marked off in tenths of a unit and the circumference marked in tenths of a unit. The students will use the wrapping function to find W(t) when the unit of measure is in decimals. Students can use the tick marks on the graphs to give decimal estimates of the coordinates of a point W(t) on the circumference of the circle given t, an arc length that is any real number. Students could finish Activity Sheet 6.2.1 for homework. You can use **Exit Slip 6.2.1** at this point.

**Teaching Strategies**

**Activity 6.2.2 Unit Circle Definition of Trigonometric Functions**

**Group Activity – Activity 6.2.2. Unit Circle Definition of Trigonometric Functions.** Direct the students to find the coordinates of the few random points marked on a coordinate plane drawn on one centimeter dot paper. Provide students with centimeter rulers and protractors. After the groups are done, ask each group to describe their approach to the problem of determining the coordinates of a point.

With students in groups, begin this activity by distributing Distribute Activity 6.2.2 **Unit Circle Definition of Trigonometric Functions.** Direct the students to find the coordinates of the random points marked on a coordinate plane drawn on one centimeter dot paper. Provide students with centimeter rulers and protractors. If you have neither, paper copies of both are downloadable for free from the internet.

You may have to enlarge the 1 cm dot paper provided with the activity on a Xerox machine if you want the paper to be a true centimeter scale. The advantage is that students can locate a point by measuring line segments with a centimeter ruler. (When you ask students to tell you the coordinates of the points they need to decide upon a unit of measure – note that telling the units when you tell the coordinates are essential. Challenge students to find two ways to locate each point on the plane. Encourage students who see the central angle in standard position formed by the point, the origin and the x axis. If needed, guide their thinking by wondering what directions you would give someone starting at the origin and wanting to end at the given point. A direction and a distance are needed. The ray from the origin to the given point can be seen as the terminal ray of a central angle in standard position. The direction can be given by measuring the central angle. The distance the point is from the origin can be seen as the length of the radius of a circle centered at the origin. You can tell students that this latter set of coordinates (radius, angle) is called polar coordinates. In the next few pages, the will be locating points using rectangular coordinates, and using the central angle on a unit circle.

After the groups are done, ask each group to describe their approach to the problem of determining the coordinates of a point. Be sure to show students that you can use the ruler to directly measure how far the point is from the y axis and how far it is from the x axis – that the length of the vertical segment from the point to the x axis and the horizontal line segment on the x axis are the y and x coordinates of the point.

Your discussion should center upon a few ideas: 1) the y coordinate of a point is the same as the directed measure of the vertical line segment drawn from the x axis to the point, 2) this vertical line segment is perpendicular to the x axis, and 3) the x coordinate is the directed distance of a line segment drawn along the x axis from the origin.

Another main idea is that coordinates are like an address for locating a point on a plane (like finding buried treasure from a map), and there are several ways to describe the directions from the origin to a point on the plane. When a circle is drawn through the point and centered at the origin, the length of the radius tells how far the point is from the origin; and the central angle in standard position that contains the point will provide the orientation. Notice that directions can be a distance and an orientation from the origin (as the crow flies) as well as horizontal and vertical distances from the origin (as you would walk on a grid of city streets).

**Differentiated Instruction** To encourage students’ curiosity and to whet their appetite for more math to come, have them do very preliminary research on polar coordinates. Direct them to research how to graph a few point in polar coordinates on paper specially marked for polar graphing. You might show them the place on their calculator mode key that allows them to graph in polar functions. Ex: on TI 84 calculator: in polar and degree mode: graph: r =s in(5$θ$) , window 0ᵒ < $θ$ <720ᵒ, $θ$ max = 720ᵒ, $θ$ step = 5, -1 < $x$ < 1, -1 < y < 1

Now students are ready to discover the relationship of the right triangle definition of the trigonometric functions to the circular definition of the trigonometric functions. By definition, we will define the trigonometric functions of any real number t as follows: cos(t) = x coordinate of W(t) on the unit circle, sin(t) = y coordinate of W(t) on the unit circle and the tan(t) = y/x .

Have students pull out the 1 cm dot paper from **Activity 6.1.8 Special Angles** on which they drew the special angles on the circle radius 10 cm. A blank copy of the page is provided if students can’t find it.( Embeddedmath.com also contains free downloadable graphs of a unit circle blank fill ins and filled in answers as well on a separate form.)

The circle is a unit circle if we use the decimeter as the unit of measure. Have students mark the point where the terminal ray of a 30° angle intersects the circle with radius 1 decimeter. Challenge students to find the y coordinate of the point by sketching a line segment from the point perpendicular to the x axis and measuring that line segment with their centimeter ruler (it should read 5 cm or .5 decimeters). – just like they did with random points at the start of the activity. They can now label the length of the vertical leg and the y coordinate of the point where the terminal ray of the angle intersects the circle. To find the x coordinate of this point, sketch the line segment along the x axis that completes the right triangle associated with 30°. Based on the triangle they drew, ask students, what is the sin 30° using the right triangle trig definition of opposite leg/hypotenuse? Point out that this ratio is the same as the y coordinate of the point of intersection on the unit circle. So we could define the sin(30°) to be the y coordinate of the point of intersection of the terminal ray of the angle in standard position that measures 30° and the unit circle. Students can check the value of sin(30°) on the calculator. Have students find the x coordinate of this point by both measuring the leg of the reference triangle on the x axis and then find the cos(30°) by finding the ratio of the measures of the adjacent leg to the hypotenuse in the reference triangle. The students can check their decimal approximation .87 decimeters or 8.7 cm with the calculator approximation of cos(30°) .



You may want to remind students of the special triangles they learned in geometry, that the sin(30°) in any 30°-60°-90° degree triangle is ½ because the sides of similar triangles are proportional.

A note about using 8.7 cm or .87 decimeters: Since we wish to work with a unit circle and the radius we have is 10 cm, we have two choices: 1) we can either regard the unit of measure as the centimeter, and then divide the coordinates by 10 in order to scale our results down to a unit circle, or 2) we can regard the unit of measure that we wish to work with as the decimeter and convert our centimeter readings to decimeters by dividing by 10 (e.g. 5 cm = .5 decimeters). Because of similarity, a third approach is to recognize that if we use a radius measured in centimeters, then the hypotenuse of the triangle is 10 cm, and the sin(30°) can be defined as the y coordinate 5 cm divided by the hypotenuse 10 cm. The unit circle definition is a specific case of sin(t ) = y/radius because in the unit circle, the radius is 1. Avoid getting bogged down in these details, but just be aware that some students will want clarification. Remember that the main goal is to have students make a connection between sin(t), the ratio of the opposite leg to the hypotenuse (which is the radius) and the y coordinate of the point .

**Activity 6.2.2 continued:** Tell students to use the same color pencil to sketch the following angles and triangles labeled in both degrees and radians: t = 150°, 5π/6, 210°,7π/6, 330°, 11π/6, and to find the coordinates of the point where the terminal ray of the angle intersects the circle. Note that these are the same points at the end of arcs that are t = 5π/6, 7π/6, 11π/6 units long. Encourage students to find shortcuts when they are ready. They will find that they do not need to measure the adjacent and opposite legs of each reference triangle. Soon enough the students will begin to use the odd and even symmetry of the points to find the coordinates of each point. Tell the students that these four angles have the same reference angle. The reference angle is the acute angle formed by the terminal ray of an angle and the x axis.

Note: it is standard use the symbol P(t) to indicate the point of intersection of the unit circle and the terminal ray of central angle of measure t in standard position. Using both P(t) and W(t) may cause unnecessary confusion. W(t) is the ordered pair of coordinates of the terminal point of the arc measuring t, where t is a real number. Since W(t) = P(t) = P(a,b), we use just the symbol W(t) to indicate the point on the unit circle associated with t.

Using a different color, have students sketch angle π/4, find the coordinates of W(π/4) by measuring legs of triangles, and then continue finding the coordinates of W(t) for t = 3π/4, 5π/4, and 7π/4. Lastly, students can use a third color to draw angle t and find the coordinates for W(t) for t = π/3, 2π/3, 4π/3, and 5π/3. For each angle, they can write in the equivalent degree measure. Ask students what is the measure of the reference angle for each of these angles? Point out that the triangles are congruent if the reference angles are equal . Why? (Because the triangles formed have congruent acute angles, are all right triangles and all hypotenuses are congruent because the hypotenuses are the radii of the same circle. The triangles are congruent by Angle-Angle-Side. Therefore we expect the trigonometric values for the 4 congruent triangles to be the same except for their signs.

Some students will be motivated to do a really artistic presentation of the beautiful symmetries in the unit circle. You could showcase their work, pointing out, perhaps, how their graphics enhance the mathematics. Here is where art and mathematics can intersect since mathematics has been called the science of patterns.

Since the decimal values they obtained for the coordinates W(t) may be too large or small depending on the accuracy of student sketches and measurements, direct the students to regard this copy as a rough draft and tell them they will create a final copy of the unit circle with the angles t and the exact coordinates for W(t) in terms of π, as directed in Activity Sheet 6.2.3. You may want to have students do Activity Sheet 6.2.3 for homework.

Throughout the activity, you will be circulating among the groups, reinforcing ideas that are leading toward a coherent circular definition of trigonometric functions. For example, you may ask students if they can double check the accuracy of their coordinates. They could confirm that x2 + y2 = 1 or they may check the reasonableness of their estimates by finding the sine or cosine of the angle on the calculator. They can also verify that $\frac{\sqrt{3}}{2}≈.866 and \frac{1}{\sqrt{2}}≈.717$ . Students may need an explanation about approximate versus exact values. You can ask whether or not .8660 = $\frac{\sqrt{3}}{2}$, for example.

Pull the class together for a whole class discussion about the relationship between the right triangle definition of trigonometric functions of acute angles and the circular definition of trigonometric functions of a real number. Students need not necessarily finish graphing all 12 of the special angles t and the coordinates of W(t) . By asking probing questions, help students define the trigonometric functions not in terms of ratios of sides in a right triangle, but rather as coordinates (or the ratio of coordinates) for W(t) on the unit circle. The class discussion should lead to a definition of the sine and cosine of a real number in terms of the x and y coordinates of W(t), the point that is the output of the wrapping function.

**Activity 6.2.2 Final Stage:** Have students draw a unit circle with the exact coordinates of these points W(t) for the special angles. This will be an important reference for them, so they should organize it in a way that makes sense to them. Just be sure the unit circle drawing includes the special angles in radians t and degrees and the ordered pair for each W(t). After the ordered pair, they may want to write the ratio of the y coordinate to the x coordinate for the tan(t). Have students make a key for their diagram by indicating which numbers indicate the radian measure of t, the cosine of t, the sine of t, and the tangent of t. Students should put this circle in a prominent place in their notebooks, possibly using plastic page protectors. Students usually find this circle to be a very useful reference page.

This is a good time to give **Exit Slip 6.2.2** that asks students to tell one thing they learned today, and one thing they still don’t understand.

**Activity Sheet 6.2.3 Working with Circular Definition of Trigonometric Functions** asks students to work with their new definition of trigonometric functions of any real number by presenting a variety of questions about circular functions and the unit circle. Problems include a) finding the trigonometric function of t , b) finding the trigonometric function of t plus integer multiples of π, given the trigonometric function of t; c) finding the trigonometric function of the quadrantal angles; (a quadrantal angle is one such as 0° or 90º, whose terminal ray lies on an axis) d) finding trigonometric functions of the angle even if the angles are not special, i.e. find the trigonometric function of any real number t; and d) using various representations of the trigonometric functions, e.g., given cos(t) = .4, find the two possible coordinates of W(t) by using the formula for a unit circle. From W(t) we can obtain the sin(t) and the tan(t).

**Exit Slip 6.2.3** can be given at this time to check if everyone understands the basic ideas of the circular definition of trigonometric functions.

**Journal Prompt 1** Explain how the right triangle definitions for trigonometric ratios are related to the circular definitions. Explain how they are different. Illustrate your explanation with the example: On a unit circle, sketch the angle 4π/3 in standard position on the coordinate plane, indicate the reference triangle, sketch the reference triangle, label the lengths of the sides of the reference triangle and find the 3 trigonometric functions of t. Students might mention the difference in the domains of the functions, for example the sin(ϴ) has a domain (0°,90°) while the sin(t) is (-∞,∞). They might note the right triangle trigonometric definitions define the functions as ratios of sides of a right triangle, but the circular definitions are in terms of the coordinates of points on the unit circle, the point being determined by the wrapping function. They might mention that a right triangle determined by the point on the unit circle, the origin, and the foot of the perpendicular on the x-axis from the point on the circle can also be used to finds the coordinates of the point W(t). Many other observations are possible.

**Closure Notes**

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In class discussion, ask the students to recount the journey of mathematical ideas that they have taken over the past 7 or 8 days: starting with right triangle trigonometry from geometry, through the introduction of radian measure, to the wrapping function that maps any real number to a point on the unit circle, and finally extending the domain of the trigonometric functions from

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**Vocabulary**

Circular definition of the trigonometric functions

Decimeter

Directed Distance

Directed length of a line segment

Period or periodic behavior

Quadrantal Angles

Reference angle

Reference triangle

Right triangle definition of trigonometric functions

Special triangles

Trigonometric functions of real numbers

Trigonometric functions of acute angles in right triangles

**Resources and Materials**

**Activities 6.2.1, 6.2.2, and 6.2.3 should be completed in this investigation**

Activity Sheet 6.2.1 Wrapping Function

Activity Sheet 6.2.2 Circular Definition of Trigonometric Functions

Activity Sheet 6.2.3 Working with the Circular Definition of Trigonometric Functions

Colored markers or pencils or pens

Compass

Embeddedmath.com also contains free downloadable unit circle forms and special angles and the trig ratios that can be filled in by students

Graphing calculator computer software with a graphing utility

Protractor

Ruler with centimeter markings

Scissors

String, twine or ribbon