**Unit 2: Investigation 4 (3-4 days)**

**Fundamental Theorem of Algebra**

**Common Core State Standards**

* N.CN.2 Use the relation i2 = -1 and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
* N.CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials. (Note that only functions with real coefficients are considered in this investigation.)

**Overview**

Investigation 4 extends the connections between the factors of a quadratic, the *x*-intercepts of the graph, and the zeros of the quadratic function to lay the foundation for the Fundamental Theorem of Algebra stating that every polynomial equation of the form *anxn + an–1xn–1 + • • • • + a1x + a0 = 0* has at least one zero within the complex numbers. The investigation will use properties of the roots of quadratic equations with complex roots to introduce and reinforce the field properties for the set of complex numbers. From previous lessons, students know that the quadratic equation of the form *0 = x2 + bx + c* with roots *r1* and *r2* satisfy the relationship that *r1 + r1 = –b* and *r1 • r1 = c*. **Activity 2.4.3** will determine if this property that holds for real zeros can be extended to hold for complex zeros.

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Determine the number of roots of a quadratic equation based on the discriminant.
* Determine the sign of the discriminant of a quadratic equation given the graph of the corresponding quadratic function, making the connection between the x-intercepts of the graph of the function and the solutions of the quadratic equation.
* Add, subtract, and multiply complex numbers, often in the context of verifying the sum and product of the zeros of a quadratic equation.
* Demonstrate that the Fundamental Theorem of Algebra is true for any quadratic function with real coefficients, i.e., any function of the form *f(x) = ax2 + bx + c* has at least one zero within the complex numbers, by finding the solutions to the equation *0 = ax2 + bx + c* and using these solutions to factor the quadratic.

**Assessment Strategies: How Will They Show What They Know?**

* **Activity 2.4.1** **Bungee Jumping Is It Safe?** establishes the need to identify the number and nature of the roots of a quadratic function.
* **Activity 2.4.2 Zeros? Real or Complex? How Do We Know?** explores the connection between the value of the discriminant, ∆, of the quadratic equation *0 = ax2 + bx + c*, given by the equation ∆ = b2 – 4ac, and the *x*-intercepts of the graph of the function *f(x) = ax2 + bx + c*.
* **Activity 2.4.3** **Complex Zeros. Do They Have the Same Properties as Real Zeros**? introduces operations involving complex numbers and quadratic equations.
* **Activity 2.4.4** **Putting It All Together** focuses on whether the Fundamental Theorem of Algebra makes sense for quadratic functions, i.e., that every quadratic function has at least one complex zero and can be factored into linear factors involving complex numbers.
* **Exit Slip 2.4.1** assesses students’ understanding of how the value of the discriminant determines the number of zeros of a quadratic function and the number of *x*-intercepts of its graph.
* **Exit Slip 2.4.2** asks students to perform computations with complex numbers, illustrate that the commutative and associative properties of addition and multiplication remain intact, and show that a complex number has a multiplicative inverse.
* **Journal Prompt 1** gives students a quadratic function and asks them how they would decide whether the graph of the function will intersect the *x*-axis or not.
* **Journal Prompt 2** ­asks students todescribe what minimum information is needed about a quadratic function with real coefficients in order to determine its corresponding equation.

**Launch Notes**

Students begin this investigation by examining a problem situation in which a quadratic function modeling the height of a bungee jumper should not have any *x*-intercepts. This investigation, which can be given as a homework assignment, leads into a study of the discriminant and whether or not the roots of a quadratic equation of the form *x2 + bx + c = 0* satisfy the property that the sum of the roots is *–b* and the product of the roots is *c*. The investigation culminates with the development of the Fundamental Theorem of Algebra with respect to quadratic functions having real coefficients.

**Teaching Strategies**

1. **Activity 2.4.1** establishes the need to identify the number and nature of the roots of a quadratic function. We want students to be able to quickly identify the nature of the roots using properties of the graph of a parabola based on its equation written in one of three forms: (1) Standard form: *y = ax2 + bx + c*, (2) Vertex form: *y = a(x–h)2 + k*, and (3) Factored form: *y = a(x–r1)(x–r2)*.

The activity, which can be given as a homework assignment, begins with the following problem: Six Flags is going to introduce a new attraction to the park: bungee jumping. They know that the height of the jumper t seconds after jumping from a height of c feet is given by the equation *h(t) = 15t2 – 90t + c*. How high does the platform from which the jumpers drop have to be to ensure their safety? Questions that will be appropriate for this include: What property of the graph of the parabola must you find to ensure the safety of the bungee jumpers? Why does the value of c in this equation represent the height of the platform? How long will it take for the jumpers to reach the lowest point on their jump? What would be the most dangerous height that you could make the platform and still keep the jumpers safe? The investigation requires students to fill in a table with values of *c* and see how close the jumper will get to the ground. Students can use graphing technology to determine how close the jumper gets to the ground by generating the graph of the function for different values of *c* or computations can be done by hand.

While there is no one lowest point that can be reached that does not hit the ground, different solution approaches to this problem will appeal to different students. Students who investigate the zeros of the function using the quadratic formula will see that the value of the discriminant will determine whether the jumper ever hits the ground, i.e., the height ever equals zero. The value of *b2 – 4ac* = 902 – 4•15•*c* = 8100 – 60*c*. In order for this value to never equal 0, c must remain greater than 8100/60 = 135 feet. Some students may decide to use the Vertex Form of the function to determine the vertex of the parabola. For the vertex to remain positive, this would mean that the y-value of the vertex must be positive. Putting the function in vertex form we obtain the equation: *y = 15(t – 3)2 + c – 15•(32).* Therefore, c – 135 > 0 and c > 135. Finally, students could also use the Factored Form of the quadratic function if they realized that the function would have to have a double root. Thus *15t2 – 90t + c = 15(x–r)2*. Solving for r, you would find that c/15 = 9 and c = 135.

The ultimate conclusion of the activity is to see that the value of the discriminant,

*b2 – 4ac,* will have to be less than 0 in order to guarantee that the jumper will not hit the ground, i.e., that there will be no *x*-intercepts.

**Journal Prompt 1**

Given a quadratic function, how would you go about deciding whether the graph of the function will intersect the *x*-axis or not. Apply your approach to the function

*f(x) = -x2 + 5x + 6*.

Students should note that the position of the vertex of the parabola and the direction of the concavity will determine whether the graph will intersect the *x*-axis. They may also answer that the value of the discriminant will also determine the number of *x*-intercepts.

1. **Activity 2.4.2** will follow up on the results of **Activity 2.4.1** that demonstrated the necessity of finding the *x*-intercepts of a quadratic function. This activity will explore the connection between the value of the discriminant, ∆, of the quadratic equation

*0 = ax2 + bx + c*, given by the equation ∆ = b2 – 4ac, and the *x*-intercepts of the graph of the function *f(x) = ax2 + bx + c*. Students will see that for ∆ > 0, there are two real roots of the form $x=\frac{-b}{2a}\pm \frac{\sqrt{b^{2}-4ac}}{2a}$ and two *x*-intercepts of the graph; for ∆=0, there is one double real root of the form $x=\frac{-b}{2a}$ and one *x*-intercept where the graph of the parabola is tangent to the *x*-axis; and for ∆< 0 there are two complex roots of the form $x=\frac{-b}{2a}\pm \frac{i\sqrt{4ac-b^{2}}}{2a}$ and no *x*-intercepts of the graph. The activity will require students to decide the number of zeros based on the discriminant and also determine the nature of the zeros and whether or not the quadratic can be factored over the rational numbers when the coefficients *a, b,* and *c* are rational numbers. Note that regardless of whether the coefficients are rational or irrational, the number and nature of the zeros is not affected. Closure for this activity would ask students to describe in writing the connection between the number and nature of the zeros of a polynomial function and the discriminant of the quadratic function. **Exit Slip 2.4.1** asks students to determine the number of the zeros based on ∆, the discriminant.

1. In **Activity 2.4.3**, students will investigate operations involving complex numbers. The introduction to the lesson will attempt to extend the properties that the sum and product of the zeros can be determined by the values of the parameters b and c in the quadratic equation x2+bx+c = 0. If not already established in previous lessons, students should know that the roots, *r1* and *r2*, of the equation *x2+bx+c = 0* that can be factored into the form *(x – r1)(x – r2) = 0*, satisfy the property that the value of *b = –(r1 + r2)* and *c = r1•r2*. Students will then solve equations that are not factorable using the quadratic formula to see if the values of *b* and *c* still represent the opposite of the sum of the roots and the product of the roots respectively. In order to do this, students must learn how to add and multiply complex numbers. An introduction explains how operations are performed with complex numbers. Following the introduction, students will solve different quadratic functions and determine whether the roots satisfy the above stated property. Students will practice performing operations with complex numbers by doing **Exit Slip 2.4.2**. This exit slip will also lead students in finding the reciprocal of a complex number, a skill that prepares students for dividing complex numbers.
2. In **Activity 2.4.4,** students will investigate whether the Fundamental Theorem of Algebra makes sense for quadratic functions, i.e., that every quadratic function has at least one complex zero and can be factored into linear factors involving complex numbers. Since the quadratic formula is applicable to all quadratic equations, students should already know that there is at least one zero for any quadratic function. Conversely, this activity also investigates the property that for every complex number of the form *a+bi*, there is a quadratic function with real coefficients for which *a+bi* is a zero. Such a quadratic function for which *a+bi* is a zero can be found by working backwards from the zero to the factors of the quadratic by applying the following properties: if *r = a+bi* is a zero of a quadratic function with real coefficients, then: (1) *(x–r)* is a factor of the quadratic function, and (2) the conjugate, *z = a–bi*, is also a zero of the quadratic function and *(x–z)* is the second factor of the quadratic function. Thus given a complex root of a quadratic equation with real coefficients, the complex conjugate forms a second root, and the product of the linear factors forms a quadratic function with the given zeros. The quadratic factor is obtained using the property that *b* is the opposite of the sum of the roots and *c* is the product of the roots of the corresponding quadratic equation. **Activity 2.4.4 Resource** is available to use with students who have not observed that r1 + r2 = -b and r1 r2 = c for real roots r1 and r2.

For example, given the zero, 3 – 2i, of a quadratic function with real coefficients, the complex conjugate would be 3 + 2i. The sum of the roots is (3 – 2i) +(3 + 2i) = (3+3) + (-2i+2i) = 6 and the product of the roots is (3 – 2i)•(3 + 2i) = 3•3 + (-2i)(2i) = 9 + (-4i2) = 9+5 = 14. Thus a quadratic function with the zero 3 – 2i is *f(x) = x2 – 6x + 14*.

**Group Activity**

In **Activity 2.4.4** students can use a Think/Pair/Share approach as they work individually filling out the table at first and then compare their answers with a partner. Then they share their answers with the whole class to form a generalization about complex roots of a quadratic equation.

**Differentiation (For Learners Needing More Help)**

If you want students to practice operations with complex numbers, students could fill in the table using hand computations instead of using a calculator. While the examples do not constitute a proof, another way to differentiate this lesson for advanced students is to ask students to compute *f(m+ni)* and *f(m–ni)* algebraically and examine the results. Students would see that the real parts of the results are always the same, and the imaginary parts are always opposites whenever the quadratic function has real coefficients.

**Journal Prompt 2**

Describe in your own words what is the minimum information you need about a quadratic function with real coefficients in order to find its corresponding equation.

Students should note that the if the *x*-intercepts of the graph of a quadratic function are given, then a quadratic equation can be found using the linear factors determined by the *x*-intercepts. They should also note that given two real zeros or one complex zero would be sufficient to determine an equation of the corresponding quadratic function.

**Closure**

Use student responses to **Journal Prompt 2** to facilitate a discussion to close this investigation.

**Vocabulary**

Complex conjugate

Converse of a statement

Conjugate

Discriminant

Factored Form

Fundamental Theorem of Algebra

Quadratic Function

Root

Standard form

Vertex form

Zero

**Resources and Materials:**

* Activity 2.4.1 Bungee Jumping Is It Safe?
* Activity 2.4.2 Zeros? Real or Complex? How Do We Know?
* Activity 2.4.3 Complex Zeros. Do They Have the Same Properties as Real Zeros
* Activity 2.4.4 Putting It All Together
* Activity 2.4.4 Resource
* Graphing Utility
* TI-84 TRANSFORM App