**Unit 5: Investigation 1 (3 - 4 Days)**

**Logarithmic Functions- Inverses of Exponential Functions.**

**Common Core State Standards**

F-IF7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F-BF-5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents**.**

F.BF.4a Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. For example, f(x) = 2(x3) or f(x) = (x + 1)/(x – 1) for x ≠ 1 (x not equal to 1).

F.BF.4b (+) Verify by composition that one function is the inverse of another.

F.BF.4c (+) Read values of an inverse function from a graph or a table, given that the graph has an inverse.

**Overview**

 Investigation 1 should review and extend concepts in Algebra 1 Unit 7. Traditionally the Algebra I Unit 7 would be covered late in the academic year, or possibly not at all. Instructors therefore should consider revisiting the Algebra I Unit 7 Investigation 1 Activities 7.1.1 – 7.1.3 and 7.2.6a which is referenced in the current Activity 5.1.1 prior to beginning this investigation.

Investigation 1 introduces logarithmic functions as inverses of the exponential functions. By recalling that the exponential function is one-to-one (or increasing or decreasing), an inverse for this function exists. The graph of the newly defined logarithmic family is then explored. Properties of the exponential function give clues as to the properties of its inverse, the logarithmic function. Fundamental rules, such as the “log of a product is the sum of the logs” are discovered in activities. Lastly, problems involving exponents and logarithms are provided.

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Graph exponential functions and determine their properties.
* Graph logarithmic functions as inverses of exponential functions, and determine their properties.
* Describe the domain of , and determine which domain values give
* Demonstrate graphically, numerically and algebraically the inverse relationship between the exponential family and the logarithmic family.
* Solve logarithmic problems by rewriting as exponential problems.
* Interpret and solve an application problem that can be modeled by an exponential or logarithmic function.
* Verify that a solution makes sense in the context of the problem.

**Assessment Strategies: How Will They Show What They Know?**

* **Exit slip 5.1.1** asks students to graph a logarithmic function and describe the domain, range, end behavior, intercepts, where increasing, asymptotes and has some questions regarding transformations of the logarithmic family.
* **Exit slip 5.1.2** has problems that use properties of logarithms to simplify equations and expressions and one that requires the inverse of a particular exponential function.
* **Journal Prompt 1**. Recall that bases of the exponential function, , are . Explain why there is this restriction on the bases for exponential function and what this also says about base restrictions for logarithms.
* **Activity Sheet 5.1.1 New Beginnings** creates a need to solve bx =k by a method other than trial and error and has the students graph an exponential function and its inverse by interchanging the table of values.
* **Activity Sheet 5.1.2 Using the Definition of a Logarithm** has students use the definition of a logarithm base b to write an expression in logarithmic form when it is in exponential form and vice versa.
* **Activity Sheet 5.1.3/5.1.3A Exploring Log Functions and Exploring with a TI-grapher** has students examine the effect of the transformations translating up and down, left and right and vertical and horizontal stretching. It will then examine the impact of those transformations on the domain of the logarithmic function family, end behavior and asymptotes. Students will reinforce the general impact of kf(x), f(x)+k, f(x + k), f(kx) and the effect of changing b. Base 10 and 2 will used for the transformation effects.
* **Activity Sheet 5.1.4** **The Product and Quotient Rules for Logarithms** will develop and provide practice using the Product and Quotient Rules of Logarithms.
* **Activity Sheet 5.1.5**. **The Power Rule for Logarithms** will provide practice using the Power Rule of Logarithms, so essential to solving equations of the form bx = k. It explores some misconceptions involving logarithms. Lastly it sets that stage for using the Power Rule to assist in solving bx = k
* **Activity Sheet 5.1.6 How High is that Stack of Paper and How Many Folds?** will return to the paper folding activity but instead have students consider cutting paper in two and stacking. Then they will practice solving equations of the form exactly and approximately. The change of base formula is included in one of the exercises.
* **Activity Sheet 5.1.7 Logarithmic Scavenger Hunt**, will have students demonstrate their understanding of rules of logarithms, logarithmic evaluation, and converting between logarithmic and exponential forms.
* **Activity Sheet 5.1.8a Consequences of Being Inverse Functions** will have student inductively postulate two fundamental logarithmic-exponential identities: and
* **Activity Sheet 5.1.8b (+) Consequences of Being Inverse Functions** will have students show and follows from the composition of the function and inverse and g(x) = and hence that g is *f*-1 so .

**Launch Notes**

This investigation begins with a hands-on activity for the students. Each student is given one piece of color paper, noting the paper is one rectangle. Folding the paper in half, the students will now have two rectangles.



Pose the question: With each fold, how many rectangles do you see? Do not count rectangles you can make by putting several rectangles together. Fill out the following table with each fold.

|  |  |
| --- | --- |
| Number of folds | Number of rectangles |
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| 3 |  |
| 4 |  |
| 5 |  |
|  |  |
|  |  |
|  |  |

Let x = the number of folds and y = the number of rectangles. Plot the points on a piece of graph paper, carefully labeling the axes and intercepts. From the table, predict the number of rectangles for 6 folds, for 7 folds, for n folds. Create the function that models this behavior and graph the function on a calculator. What properties of the function are illustrated by the graph? Students should state domain and range, comment on x and y intercepts, and be able to explain why there cannot be an x-intercept. Does this function have an inverse? Students could respond that the function does have an inverse because the exponential function is one-to-one, or the exponential function is increasing, or because it passes the horizontal line test. Though there can only be an integral number of folds ask students to consider the abstract function and its graph when we draw a smooth curve through the points with integral inputs. They did this in algebra 1 with some linear function applications.

By creating an x y table and graphing the function , students should be able to answer questions about a) the domain of the function b) the range of the function c) intercepts, both *x* and *y* and d) end behavior. This function should be familiar. They met it in Algebra 1.

How many times can you fold a piece of paper? It was long believed it could be done at most 8 times, but in 2002, Britney Gallivan, a high school junior, demonstrated it could be done 12 times. See <http://en.wikipedia.org/wiki/Britney_Gallivan>.

You can also watch <https://www.youtube.com/watch?v=KRAEbotuIE> which taped the paper-folding of a piece the size of a football field for MYTHbusters Folding Paper Seven Plus Times.

What if we want to know the number of folds that produce 512 rectangles? What might we do to answer this question?

In this investigation, students will define the logarithmic functions as the inverses of the exponential functions and discover some of the properties of the logarithmic function family based on properties of the exponential function family.

Note we are using the agreement that inverse means inverse function.

**Teaching Strategies**

**In Activity 5.1.1** **New Beginnings,** following the launch activity, students return to a scenario seen in Algebra 1 Activity 7.2.6 but in a new context. This time a young man and his uncle are exploring saving for college. The problem will remind them that in Algebra 1, to solve an equation such as 27 = 2x they had to use trial and error along with technology to obtain an approximation. It was tedious and time consuming and hopefully they will now desire a more efficient and direct approach to solving such equations. The launch suggests that the existence of an inverse function for may be a path they can follow. They will revisit the table created during the launch and graph the function using technology. The tables given may be filled in two ways: (1) by considering powers of 2 and/or (2) by looking at the intersection of the horizontal line and . Initially the first technique works easily, but when given values of that are not integral powers of 2, the second technique must be used. Do several examples to illustrate this.

Describe x in words,without concern for a specific value should lead to the statement that *“x is the exponent to which 2 is raised to produce ”.* Building along these lines leads to the natural development of the inverse function, . Note that the base of the exponential function is the same as the base in the logarithmic function when dealing with a pair of inverse functions.

**Group Activity –** Students should now work in small groups of size 2 or 3 to complete **Activity sheet 5.1.1** and then discuss the relationship between the two functions. The group recorder should submit a sheet with the group’s written observations. Plan the group membership before class to ensure effective dialogue within each group.

**Differentiated Instruction (For Learners Needing More Help)** Have students create tables of values for exponents, say , with several positive values of x, several negative values and zero. Next, interchange the two columns, and graph this new function. How is this new function related to the original function? Using the table of values, rewrite exponential equations into logarithmic equations.

**Differentiated Instruction (Enrichment)** While it may not be difficult to answer what is equal to, now think about . Is this possible to do, that is, does this number exist? Is it rational? Irrational? How does our graph help us think about this question? Can we approximate it roughly or find the two integers closest to its value? What other mathematical function’s output is often approximated with a closest whole number smaller than it and a closest whole number larger than it?

The background and history of logarithms can be considered before **Activity 5.1.2.** See <http://www-history.mcs.st-andrews.ac.uk/Biographies/Napier.html> for an excellent summary of the life of John Napier as well as his discovering/creation of logarithms. In particular,

“Napier's discussion of logarithms appears in *Mirifici logarithmorum canonis descriptio* in 1614. Two years later an English translation of Napier's original Latin text was published, translated by Edward Wright. In the preface of the book Napier explains his thinking behind his great discovery (we quote from the English translation of 1616 of the original Latin of 1614):

*Seeing there is nothing* (*right well-beloved Students of the Mathematics*) *that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances. And having thought upon many things to this purpose, I found at length some excellent brief rules to be treated of* (*perhaps*) *hereafter. But amongst all, none more profitable than this which together with the hard and tedious multiplications, divisions, and extractions of roots, doth also cast away from the work itself even the very numbers themselves that are to be multiplied, divided and resolved into roots, and putteth other numbers in their place which perform as much as they can do, only by addition and subtraction, division by two or division by three.”*

To this end, Napier created this relationship between logarithms and exponentials which changes two difficult operations, multiplication and division, into the two simpler operations of addition and subtraction via the definition.

 if and only if

From this definition, we now see that

 since

 since

 since

Emphasize the relationship between the numbers found above to the numbers in the tables as well as the graphs created in **Activity 5.1.1**.

Have students note that

 since

Emphasize that log base 2 of y is the exponent one raises 2 to in order to get y.

**Activity 5.1.2** **Using the Definition of a Logarithm** can be now be used in class or for homework. It will reinforce the definition

 if and only if

And more generally

 logb y = x if and only if bx = y

and will continue to have students think that log base b of y is the exponent we raise b to in order to get y.

**Journal Prompt 1** Recall that bases of the exponential function, , are . Explain why there is this restriction on the bases and what this also says about base restrictions for logarithms. Students might respond f(x) = 1x = 1 for all x and the graph is therefore a horizontal line and thus describes a linear function. Also f(x) = 0x = 0 for all x≠0 and the graph is therefore a horizontal line with a hole. For a negative base, say -2, (-2)2 = 4 while (-2)3 = -8 so we lose continuity or a student might mention (-2)0.5 is not real and the domain would no longer be all reals. The desire for an exponential function is to have a curve that is always increasing or always decreasing, not constant. Since each exponential function has a base and that base must correspond to the base for its

logarithmic family inverse function, the logarithmic family cannot have a base of 1 or a base that is nonpositive.

**Activity 5.1.3 and 1.5.3A Exploring Log Functions (With Geogebra) and with the TI-Grapher** will have students use technology to study the effect of transformations on the graphs of the logarithmic family. It can be an individual or pair or group activity. This is the first function family students have studied that does not have a domain of all reals or whose domain is not restricted because of either a root or a quotient in the defining equation of the function. It would also be nice to really help students appreciate that though always increasing, logarithmic family members grow really slowly. If the weather is nice a trip to the football field might be nice, but if not then maybe a football field can be displayed and students asked to consider the logarithmic function just from the point (1, 0). Let the start of the football field be 0 and the first 10-yard line be 10, next 10-yard line be 20. We can locate (1,0) at the 1-yard line on the side-line 1 yard, (10,1) at the 10-yard line, 1 yard into the field, and at the 100-yard line locate (100, 2) two yards from the side line. Can they picture that (1000,3) would need 10 football field laid one after the other? All that horizontal distance to just go vertically 1 unit ---in this case 1 yard. **Exit Slip 5.1.1** can be used any time after **Activity 5.1.3** has been completed.

**Group Activity 5.1.3** Students can be paired or placed in groups of three and work together to study the effects of vertical shifts, horizontal shifts, vertical and horizontal compressions and reflections and the impact they have on the domain, end behavior, asymptotes, and intercepts.

Student can now justify two special rules for logarithms that will be developed and used in **Activity 5.1.4,** **The Product and Quotient Rules for Logarithms.** The Product Rule can be done large group and the Quotient Rule can be done in small group or as a whole class activity. Begin by having students recall the two properties of exponents, and . If needed, do examples to recall these important properties and remind them they reviewed exponents in Unit 4. **Activity 5.1.4** will prompt students to inductively discover the following rule for base 10

.

They will then rewrite some expressions and repeat the procedure the Quotient Rule. Students should be encouraged to recognize that proving it for base 10 is not the same as proving the rule for base b and hopefully they can then discuss how the steps in their proof would have to be adjusted. Both Activity 5.1.4 and 5.1.5 have a problem that will need students to take the logarithm the of both sides of an equation. If a = b and a > 0 and b > 0, log a = log b. Students need to discuss the reasonableness of this assertion. The logarithm of a number is unique since we are dealing with a function so if a and b represent the same number the logarithm of a must be the same number as the logarithm of b.

**Group Activity 5.1.4** Show step by step that

**Differentiated Instruction (For Learners Needing More Help)** Consider concrete examples, such as = 2 which can also be written as

What restrictions need to be placed on b, A and B in the two properties below?

**Activity 5.1.5 The Power Rule for Logarithms** explores the Power Rule for Logarithms so essential in solving bx = k and provides some practice using the law itself.

**Activity 5.1.6** **How High is that Stack of Paper and How Many Folds?** will revise the paper folding activity and then use the Power Law to assist solving an equation of the form bx = k. Students should be asked to consider the reasonableness of: If a = b then log a = log b where a > 0 and b > 0. Equivalent equations are obtained. The change of base formula is included in the exercises and need not be completed by all students. Exit slip **1.5.2** can be used after this activity has been discussed.

**Activity 5.1.7** **Logarithmic Scavenger Hunt** provides an opportunity for students to review all the properties and definition of logarithms so is a good review activity. Because it does not focus on just one or two of the properties of logarithms it requires students to think about all properties and forms they have studied. Students can be assigned in pairs, groups or individuals. The student activity file contains the problems you will need to print and post on the walls of the classroom and a student answer key.

**Activity 5.1.8a** **Consequences of Being Inverse Functions** is for all students. It has students discover by inductive reasoning that and when and g(x) = . They already met one of these observations in Activity 5.1.2. But there their attention was probably focused on using the definition if and only if . Remind them that they have not proved the observation. STEM intending students can then justify using composition of the functions that these two observations are indeed always true with of course appropriate domain restrictions. In **Activity 5.1.8b (+) Consequences of Being Inverse Functions,** students will show and follows from the composition of the functions and . This activity reinforces that these two properties are consequences of other definitions and concepts and not just a rule they must remember. Deductive reasoning is in Algebra not just Geometry!

**Differentiated Instruction (Enrichment)** Activity Sheet 5.1.8b+ provides an opportunity for students who explored composition in unit 1 to apply that knowledge.

**Differentiated Instruction (Enrichment)** When we write and , there are restrictions on *x* based on domains of functions and their inverses. Carefully rewrite the rules and by including the restrictions on *x.*

**Closure Notes**

This unit lays the foundation for defining the logarithmic function base b as the inverse of the exponential function base b. We know by looking at the graph of an exponential function, an inverse function will exist. The next question is then, how do you explicitly find the inverse. Students start to find the inverse algebraically, as they did in unit 1 by replacing f(x) with y, and solving for x. But quickly they discover that they have no means for solving for x explicitly. A *definition* comes to the rescue.

While completing the activities for this investigation students met the following definitions and properties.

If are positive real numbers, , and is any real number, then

1. , for any real number

A group activity to consider is one where the class is divided into groups of 2-3 students and each group is given the properties from the list above. In the group, students are then instructed to create concrete examples with numbers to illustrate the properties, 2-3 for each property. One group member from each group could then put up the group examples for property *a* on the board. Once examined, students can then move on to property b.

**Vocabulary**

base

exponent

exponential function

function

inverses of functions

logarithm

logarithmic functions

one-to-one function

Power

**Resources and Materials**

**All activities should be completed except 5.1.8b (+) which is for Stem intending and which may be omitted. Activity 5.1.7 is a review activity so it may be omitted but is has students use all the logarithm laws that were developed so students need to be able to distinguish between the laws. The change of base exercise in Activity 5.1.6 may also be omitted.**

Activity Sheet 5.1.1 New Beginnings

Activity Sheet 5.1.2 Using the Definition of a Logarithm

Activity sheet 5.1.3/5.1.3A Exploring Log Functions (With Geogebra) and with the TI-Grapher

Activity sheet 5.1.4 The Product and Quotient Rules for Logarithms

Activity sheet 5.1.5 The Power Rule for Logarithms

Activity Sheet 5.1.6 How High is that Stack of Paper and How Many Folds?

Activity Sheet 5.1.7 Logarithmic Scavenger Hunt

Activity sheet 5.1.8a Consequences of Being Inverse Functions

Activity Sheet 5.1.8 b (+) Consequences of Being Inverse Functions

<http://en.wikipedia.org/wiki/Britney_Gallivan>

<http://www-history.mcs.st-andrews.ac.uk/Biographies/Napier.html>

<https://www.youtube.com/watch?v=KRAEbotuIE> which taped the paper-folding of a piece the size of a football field for MYTHbusters Folding Paper Seven Plus Times.

Calculator

Colored paper

Graphers

Internet to read about Napier

Graph paper for activity 5.1.1