**Activity 5.5.2: Modeling the Population of the United States**

In the last activity, you determined that the exponential function $P=3.929e^{0.3008t}$ was a good model for the population of the United States for the years close to 1790 ($t=0$). In later years such as 1950 ($t=16$), the original exponential function was no longer a good representation of the population. The population still appeared to be growing exponentially in later years but with a lower growth rate.

Let’s develop a piecewise function to model the early exponential growth and the later exponential growth. Here is the population data.

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| **Year** | **Population (in millions)** |
| 1790 | 3.929 |
| 1800 | 5.308 |
| 1810 | 7.240 |
| 1820 | 9.638 |
| 1830 | 12.866 |
| 1840 | 17.069 |
| 1850 | 23.192 |
| 1860 | 31.443 |
| 1870 | 38.558 |
| 1880 | 50.156 |
| 1890 | 62.948 |

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| **Year** | **Population (in millions)** |
| 1900 | 75.996 |
| 1910 | 91.972 |
| 1920 | 105.711 |
| 1930 | 122.775 |
| 1940 | 131.669 |
| 1950 | 150.697 |
| 1960 | 179.323 |
| 1970 | 203.185 |
| 1980 | 226.546 |
| 1990 | 248.710 |
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**Part 1 – The First Piece**

You have already created an exponential function representing the population for the early decades.

$$P=3.929e^{0.3008t}$$

Graph the function and the data points.

For which values of$ t$ is this exponential function a good fit?

$$0\leq t< \\_\\_\\_\\_\\_\\_\\_$$

You have just completely defined the first part of the piecewise function. Fill in the missing $t$-value below.

$$P=\left\{\begin{matrix}3.929e^{0.3008t } for 0\leq t<\\_\\_\\_\\_\\_\\_\\_\\_\\_\\ \\ a∙e^{kt } for \\_\\_\\_\\_\\_\\_\\_\\_\leq t<21\end{matrix}\right.$$

**Part 2 – The Second Piece**

Now you have to find an exponential function that models the rest of the data. The plan is to choose two points, fit an exponential function to those two points, and then graphically determine if the function is a good fit for the second part of the data. This involves solving a system of non-linear equations so let’s try it a couple of times. You will get practice solving systems and you will have a couple of exponential models to choose from.

***First Attempt – 1930 and 1940***

1. Use the data from 1930 and 1940 to write two ordered pairs of the form $(t,P)$.
2. Using each ordered pair, substitute $t$ and $P$ into the equation $P=a∙e^{kt}$. This will give you two equations with two unknowns ($a$ and $k$). Label one Equation 1 and the other Equation 2.

You have prior experience solving systems of linear equations. One technique that you used was substitution. You solved one equation for one variable and then substituted into the other equation. You can also use substitution with non-linear systems! Let’s try it.

1. Solve Equation 1 for $a$.
2. Substitute this expression into Equation 2 and simplify.
3. You should now have a one variable equation. Solve this equation.
4. You have the value of the parameter $k$. Substitute this value in to either Equation 1 or Equation 2 and determine the value of parameter $a$.
5. You have values for both parameters. What is the exponential function?
6. Graph this function with the original data points. How well does this function match the data points?

***Why?***

1. When using substitution to solve a system of 2 equations in 2 variables, you solve one equation for one variable and substitute into the other equation. Could you have solved the system if you chose to initially solve for $k$ instead of $a$? Why were you directed to solve for $a$ rather than $k$ in question 3?
2. Earlier in this unit, you saw that exponential equations could be written as $y=a∙b^{t}$ or $y=a∙e^{kt}$. The variables $y$ and$ t$ are the same in both versions of the equation and the parameter $a$ is the same in both versions. Recall $b$ is the growth factor of the exponential function. How is $k$ related to $b$?

Using the population from 1930 and 1940, what is the growth factor $b$?

Using the growth factor, *b*, determine the value of $k$.

In question 5, you found $k$ through substitution and solving a one variable equation. Does the value from question 5 match the value you just found?

***Second Attempt – 1940 and 1950***

Try this with a different set of data points. You are going to find another exponential function using the data from 1940 and 1950. Once you have a function, graph the function with the data points to determine if the function better fits the data.

1. Use the data from 1940 and 1950 to write two ordered pairs of the form $(t,P)$ and two equations of the form $P=a∙e^{kt}$. Solve the system of equations.
2. What is the exponential function?
3. Does the graph of this function fit the data? Is the fit better or worse than the exponential model from the first attempt?

**Part 3 – The Piecewise Function**

1. You can now write the entire piecewise function. Fill in the missing parts below.

$$P=\left\{ \begin{matrix}3.929e^{0.3008t } for 0\leq t<\\_\\_\\_\\_\\_\\_\\_\\_\\_\\ \\ for \\_\\_\\_\\_\\_\\_\\_\\_\leq t<21\end{matrix}\right.$$