**Activity 5.1.1 New Beginnings**

This story is a modern day adaptation of the King’s Chessboard, a story with a mathematical twist from ancient India.

On New Year’s Eve, December 31, Sam is attending a celebration with his family, including aunts, uncles, grandparents, cousins, and more! Sam’s favorite uncle knows that Sam is planning to go away to college next year. His uncle has come to the party with ten crisp new $100 bills to give to Sam to help finance his education. As the evening wears on, Sam and his uncle begin chatting. Sam’s uncle decides to have a little fun with Sam.

**Uncle Charlie:** Sam, I want to help you out with paying for college. I’ve got two options and you can pick whichever one you want.

**Sam:** Thanks! I have been a little worried about paying for school. What are the options?

**Uncle Charlie:** Well, here is $1000.

He pulls out the ten brand new $100 bills and puts them in front of Sam. Sam’s eyes open wide – he had never seen that much money in one place before!

**Sam:** Whoa….

**Uncle Charlie:** And here is the second option.

Uncle Charlie proceeds to toss a dull, dirty penny on the table. Sam looks confused but his uncle begins to explain.

**Uncle Charlie:** This penny symbolizes an agreement. If you take the penny, I will give you double that tomorrow, the first day of the new year. Then on the second day of the new year, I will give you double what I did on the first day. I will continue with this pattern for 64 days.

Sam starts to reach for the $1000 but then thinks more carefully about the penny. On day 1, he would receive 2¢. On day 2 he would get 4¢, on day 3 he would get 8¢, on day 4 he would get 16¢, and so on. This would be less than $3 total in the first week! But Sam had paid attention in math class. This was not a linear pattern. Why? \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ This is a(n) \_\_\_\_\_\_\_\_\_\_\_\_\_ pattern! Did his uncle really know what he was offering? This deal was too good to be true! Sam smiled slyly and slid the penny toward him. Why? \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

Uncle Charlie, lived up to his end of the deal for as long as possible.

1. How much money did Uncle Charlie give Sam on the 21st day of the year?
2. How much money was Uncle Charlie supposed to give Sam on the 64th day of the year?
3. As you saw in the paper folding activity and in Algebra 1, this type of pattern can be represented by an exponential function. In this example, if you know the day of the year, you can substitute that value into the exponential function and then figure out how much money Sam’s uncle gave him that day. But what if the question is reversed?

Describe the process you would use to determine the answer to this question and then put that process into action to answer the question.

1. On what day of the year did Sam’s uncle give him 2097152¢ or $\_\_\_\_\_\_\_\_\_\_\_\_\_ ?
2. On what day of the year did Sam’s uncle give him $335544.32?
3. On what day of the year did Sam’s uncle give him 2,147,483,648¢ or $ \_\_\_\_\_\_\_\_\_\_\_\_\_?

Your process for determining the day on which Sam received that amount of money probably involved some calculations and a little trial and error to get to the correct day. If you use the idea of inverse functions, you can figure out the day without trial and error. Let’s start with graphing an exponential function and then working towards developing the inverse function.

**Graphing the Exponential Function** $f(x)=2^{x}$

1. Complete the following table.

|  |  |
| --- | --- |
| $$x$$ | $$y=f(x)$$ |
| -2 |  |
| -1 |  |
| 0 |  |
| 1 |  |
| 2 |  |
| 3 |  |

1. On the graph paper provided, graph the function $f(x)=2^{x}$. On your graph, clearly indicate the six points from the table. (Hint: You might want each unit on the axes to represent $\frac{1}{2}$ in order to have an easier time graphing the fractions.)
2. By looking at your graph, how can you tell that $f(x)=2^{x}$ has an inverse (function)?
3. What is the domain of the exponential function $f(x)=2^{x}$?
4. What is the range of the exponential function $f(x)=2^{x}$?
5. State the asymptotes of graph of the function $f(x)=2^{x}$. Make sure you give an equation for the asymptote and also state whether it is a vertical or a horizontal asymptote.

**Graphing the Inverse of the Exponential Function** $f(x)=2^{x}$

1. Do you remember how functions and their inverses are related? Use the table below to list six ordered pairs **on the graph of the inverse function**. Next to the table, describe how you obtained these ordered pairs.

|  |  |
| --- | --- |
| $$x$$ | $$y$$ |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |

1. On the graph paper provided, graph the six points from the table.
2. Keep in mind that inverses are reflections over the line $y=x$. Graph the line $y=x$. Use the points that you have already graphed as well as the reflection property of inverses to completely sketch the graph of the inverse of $f(x)=2^{x}$. (Hint: Check that your graph is correct by folding your paper on the line $y=x$.)
3. What is the domain of this inverse function that you just graphed?
4. What is the range of the inverse function?
5. State the asymptotes of graph of the inverse function. Make sure you give an equation for the asymptote and also state whether it is a vertical or a horizontal asymptote.

**Writing an Equation for the Inverse of the Exponential Function** $f(x)=2^{x}$

You know the inverse of $y=2^{x}$ exists and you even know what the graph of the inverse looks like. Now it is time to get an equation for the inverse of an exponential function. Remember that to get the inverse function, you swap the $x$ and $y$ values in the original equation (that’s where the idea of reflection across the line $y=x$ comes into play) and then solve for $y$.

Original Exponential Equation: $y=2^{x}$

Inverse Equation: $x=2^{y}$

Notice the location of y in the inverse equation! The **y is in the exponent** and you currently do not have a way to solve for y and get it out of the exponent. This is where the logarithm function comes in.

**Definition of a Logarithm(Base 2)**

$y=log\_{2}x$ is equivalent to $x=2^{y}$

The equation of the inverse of $f(x)=2^{x}$ is $f^{-1}(x)=log\_{2}x$.

 Define the inverse of each function by rule.

1. f(*x*) = 8*x.*
2. g(*x*) = log3*x*
3. p(*x*) = 10*x*
4. r(*x*) = log5*x*