**Unit 2: Investigation 3 (3 Days)**

**Complex Numbers**

**Common Core State Standards**

N.RN.3 Explain why the sum or product of two rational numbers is rational and the sum of a rational and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

N.CN.1 Know there is a complex number *i* such that *i*2 = -1, and every complex number has the form *a* + *bi* with *a* and *b* real.

N.CN.2 Use the relation *i*2 = -1 and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers

N.CN.7 Solve quadratic equations with real coefficients that have complex solutions.

**Overview**

Students will learn that the desire for closure of a set of numbers over a given operation motivates the invention of new types of numbers and that complex numbers are the set of numbers that brings closure to the inverse operation of squaring. Students may wonder how we can just make up a new type of number. They will learn that negative integers were invented to address the lack of closure of the natural numbers over subtraction. Rational numbers are a response to the lack of closure of division over the integers. Students will investigate closure of sets of numbers under the four basic operations: add, subtract, multiply and divide, and the square root operation. Students also explore why the sum and product of an irrational and a rational is a member of the irrationals (provided neither of the factors is 0). After students are introduced to the pure imaginary numbers and investigate *in* (*n*, a natural number), they are introduced to complex numbers. Students will practice the operations of finding the conjugate of, adding, subtracting and multiplying complex numbers. STEM-intending students will learn about complex conjugates and the division of complex numbers. Finally, students will use complex numbers to solve any quadratic equation with real coefficients over the complex numbers.

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Identify types of numbers: natural (counting), whole, integers, rational, irrational, real, complex.
* Determine whether a set of numbers is closed under an operation.
* Determine whether or not a number, or sum or product of numbers is rational or irrational.
* State the definition of *i*.
* Express the square root of a negative number in terms of *i.*
* Evaluate i*n* for any natural number *n.*
* Identify the real and imaginary parts of a complex number, using the a+bi notation.
* State the conjugate of a complex number.
* Add, subtract, and multiply complex numbers.
* Solve quadratic equations that have complex solutions.
* State the two solutions in a+bi form.
* Given a single complex solution to a quadratic equation, give the other solution – the complex conjugate. This is true only if the quadratic equation has coefficients that are real numbers.

**Assessment Strategies: How Will They Show What They Know?**

* **Exit Slip 2.3.1** assesses students understanding of the closure of sets of real numbers.
* **Exit Slip 2.3.2** asks students to simplify expressions involving complex numbers.
* **Exit Slip 2.3.3** asks students to solve a quadratic equation with imaginary solutions.
* **Journal Prompt 1** asks students to draw a Venn Diagram showing which sets of numbers are subset of others and which are disjoint. The sets of numbers include Natural (counting), Whole, Integers, Rationals, Irrationals, Reals, and Complex.
* **Journal Prompt 2** asks students to agree or disagree with Kronecker’s statement “Integers were created by God, all else is the work of man”, and explain their reasoning.
* **Activity 2.3.1 Closure and Sets of Numbers**
* **Activity 2.3.2 Imaginary** **Numbers** is a series of problems that lead students through some operations with the pure imaginary number *i* and constant multiples of *i*
* **Activity 2.3.3 Complex Numbers**
* **Activity 2.3.4 Complex Solutions to Quadratic Equations** will direct students to find the complex solutions to equations that did not have real solutions interspersed with a few equations that do have real solutions

**Launch Notes**

The launch sets the stage for why complex numbers were developed. Ask students if they recall being told something that they couldn’t do in arithmetic, but they later learned that it was possible. When students first learned to subtract numbers, they may have been told that they can’t do a problem like 3-12. Ask them what it was like when they first learned to divide numbers like 12 and 3. They may have been told that 12 can’t be divided by 7, and then someone may have added “at least not evenly”, meaning that 12 coins divided among 7 friends can’t be done so that everyone has their fair share. Then we learned that quotients need not be whole numbers. What is it about cookies that enable us to share 12 cookies among 7 friends? (Answer: you can take a fractional part of a cookie, but not a coin). So with division, we needed to invent fractions. It seems as if humans create new sets of numbers as needed. So many distinct cultures created the counting numbers that one may suspect that counting is part of our essence as a human being. Psychological studies on infants 3 days old prove that babies are born with a sense of number. Babies show by their reaction to novel stimuli that they can distinguish between 1 item, 2 items and many items. Ask students how toddlers count: “1, 2, 3, 100”, or “1, 2, skip a few”. Conclude with the quote from Ludwig Kronecker, “The integers were created by God, all else is the work of man.” Tell students to keep Kronecker’s words in mind as they work through Investigation 3, because you will ask them later to write in their journal what they think he meant and whether they agree. There is no correct answer to the question about whether numbers were discovered or created, as many different philosophies of mathematics exist.

**Teaching Strategies**

Distribute the first two pages of **Activity 2.3.1 Closure and Sets of Numbers** before the launch. Students will be tested on their number sense, distinguish between decimal approximations and exact values for repeating decimals and irrational numbers, and have an opportunity to disagree and argue their answers with other students. After discussing student answers to the first two pages of the activity, begin a discussion about sets of numbers. See the **Activity 2.3.1 Teacher Notes** for discussion and storytelling ideas.

After you teach students about the types of numbers: (natural (or counting), whole, integer, rational, irrational, and real), have students work in groups on pages 3-7**.** At the conclusion of the activity, you will need to bring the class together to talk about the rational and irrational numbers, especially why a rational number plus an irrational number is irrational. Give an example of an everyday proof by contradiction. For example we assume a criminal is innocent. When we have seen enough evidence contradicting the assumption of innocence, we reject that assumption and prove the criminal guilty. Another example: Jon is a good friend. I hear that he is talking behind my back. That is a contradiction, because good friends don’t talk behind each other’s backs. Therefore, my original assumption that Jon is a friend must be false.

**Differentiated Instruction (Enrichment)**

Students can prove that there are the same number of counting numbers as whole numbers, the same number of whole numbers as integers and rationals using the idea of a one to one correspondence. The proof that there are more reals is also accessible using a proof by counterexample.

You can give the **Exit Slip 2.3.1** a set of true/false statements about numbers at the conclusion of **Activity 2.3.1.**

**Differentiated Instruction (Enrichment)**

**Activity 2.3.1** teaches students a method for changing repeating decimals to ratio form. For example: Write .181818… as the ratio of two integers.
Let N = the number with repeated decimal = .181818….

Since the repeating pattern occurs every two digits, multiply N by 102.

100N = 18.1818….

Then subtract N (the infinite part) from both sides.

100N = 18.1818….

-N - .1818…

99N = 18.0000…

Solve for N: N=18/99 = 2/11

**Activity 2.3.2 Imaginary** **Numbers** is a series of problems that lead students through operations with the pure imaginary number *i* and constant multiples of *i.* Before you distribute **Activity 2.3.2**, ask students to give examples about whether the set of real numbers is closed under the operation “take the square of”. Then ask if the real numbers are closed under the inverse operation “take the square root of”. You might want to point out that squaring and extracting roots are unary operators. Ask students to name as many binary operation and as many unary operations as they can.

The desire for closure of the reals under extracting roots motivates the need for complex numbers, but first we need to define the foundational number for complex numbers ‘i’, a pure imaginary number that is, by definition, the solution to the equation *x*2 = -1. Then have students help you reason from basic the definition of i as the number that when squared gives -1 to the other powers of i. Have them develop the pattern of i, -1, -i, 1, i, -1, -i, … Use the fact that i4=1 to simplify, say, i38 by dividing 38 into 9 groups of 4 with 2 remaining: i38= i4(9)+2= (i4)9 times i2= 1 times -1 that equals 1. Observe that the remainder (when the exponent is divided by 4) is what is important. Encourage students to find a short cut if they can, rather than you telling them “tricks.” Then demonstrate to students that $\sqrt{-3} \sqrt{-3} $does not equal$\sqrt{(-3)(-3)}$. The rules for roots of real numbers are not necessarily the same as those for imaginary numbers.

**Journal Prompt 1** (Assign after **Activity 2.3.2**)

Have students draw a Venn diagram showing which sets of numbers are subsets of others and which are disjoint. Use the sets: Natural (N), Whole (W), Integer, (I), Rational (R), Irrational (IR), Real, Imaginary, and Complex.

Students should have a diagram that has the irrationals (IR) set apart from the rationals (R), with no overlap between them, and a big rectangle around both for the reals. Within the rationals (R), N is a subset of W, which is subset of I, which is a subset of R. The imaginaries should have a rectangle around it. The reals and imaginaries should be in disjoint rectangles with a rectangle around both of them representing the complex. The intersection of sets of real numbers is addressed in Question B7 in **Activity 2.3.1.**

R

I

W

N

IR

Imaginary

Numbers

Real

Numbers

Complex Numbers

**Differentiated Instruction (Enrichment)**

**Activity 2.3.2** challenges students to find a “proof” that 1 = -1 that you can find by doing an Internet search for Fallacies involving i. The errors in these “proofs” rest on the fact that $\sqrt{a}∙\sqrt{b}=\sqrt{ab}$ only if aand *b* are positive numbers or zero.

Now that students understand the imaginary number *i,* they will learn that complex numbers are those numbers that have two parts: a real part‘a’ and an imaginary part ‘b’. The concept of a+bi representing a single number may seem uncomfortable to students, so you can liken complex numbers to fractions that have two parts a/b, a numerator divided by a denominator, and we all understand fractions represent one number.

**Activity 2.3.3a Complex Numbers** (+) is for STEM-intending students since it also includes the complex conjugate and division of complex numbers. Have students work in groups on the activity, giving them the opportunity to reason through the problems themselves, while you circulate and provide support as needed. Students will need to learn about rationalizing single term and two termed denominators for real numbers, first. Show them a few problems such a rationalizing $\frac{1}{\sqrt{3} }, \frac{2}{\sqrt{6} } \frac{\sqrt{3}}{\sqrt{7} } $. Have them guess how to rationalize$\frac{1}{2+\sqrt{3} }$. Many students will guess to multiply the numerator and denominator by$2+\sqrt{3}$, or they will say, “square the denominator.” Have them try this idea, see that it doesn’t work, then ask them if they can think how to use the fact that (a-b)(a+b) = a2-b2 to get the square of both terms in the denominator. Then have them do a variety of rationalizing denominators when the denominator has two terms. For example, use$\frac{1}{\sqrt{5}+\sqrt{3} }, \frac{5}{1-\sqrt{6} } \frac{2-\sqrt{3}}{2-\sqrt{7} }$. Be sure that no one “cancels” the two’s in the last example.

**Group Activity**

**Activity 2.3.3a** is designed for students to complete in groups.

**Activity 2.3.3 Complex Numbers** is an alternative to **Activity 2.3.3a.**  If you are going to have students finish **Activity 2.3.3** for homework, be sure you have them do at least one or two problems in class from each general category of problems in the activity. If time permits, give them **Exit Slip 2.3.2** before you send them on their way to finish **Activity 2.3.3** for homework. Otherwise, use the exit slip to assess how well they learned the material by doing their homework.

**Activity 2.3.4 Complex Solutions to Quadratic Equations** begins by asking students to work in groups to solve an equation that has no real solution. For example, find at what time the projectile modeled by h(t) = -16t2 + 48t will reach 100 feet. You can guide their work with questions you ask each group as you walk around the room. At least some students are likely to see the negative number in the square root when they use the quadratic formula, but instead of just saying “no real solution” as they did in Investigation 2, they can now write the complex solutions in a+bi form. Have a short discussion about what “no real solution” means in this context, and show that the graph of real numbers on the axes does not show the complex solutions. One needs a complex plane (not a real plane) to graph complex numbers. Students can finish the activity sheet with their group in class or individually at home.

**Activity 2.3.4** directs students to find the complex solutions to equations that do not have real solutions interspersed with a few equations that do have real solutions. Students will write the complex solutions in a+bi form. Students will check the solutions for a few equations by substituting the complex solutions back into the original equation. Students will be prompted to notice that complex roots always come in conjugate pairs, but this is true only if the coefficients of the quadratic are real numbers.

Give **Exit Slip 2.3.3** at the end of class after students complete **Activity 2.3.4**.

**Differentiated Instruction (For Learners Needing More Help)**

Some students need to review rounding decimal expressions such as .33…. to .3 and .66… to .67. Others may need help writing fractions as decimals. Explain that the fraction bar means “division”: 6/2 means 6 divided by 2. Then review long division for fractions that are both more and less than 1. Also remind students how to rewrite decimals as fractions. If a student can say a decimal number in words, (that involves identifying the place value of the last digit in a terminating decimal), they can write the decimal as an integer divided by a power of 10.

Ex: 23.456 is twenty-three and four hundred fifty six thousandths = 23 and 456/1000. Then simplify the fraction. Reading and writing a decimal correctly in words is an important financial skill necessary for writing checks.

**Differentiated Instruction (Enrichment)**

**Activity 2.3.3a Complex Numbers** includes problems about the complex conjugate and division of complex numbers.

**Journal Prompt 2** Ask students to agree or disagree with Kronecker’s statement “Integers were created by God, all else is the work of man.” Explain.

**Closure Notes**

In addition to learning the important content about what are imaginary and complex numbers, this was a very theoretical and philosophical investigation. Some students will appreciate that mathematics is much more than number crunching. All students will begin to see that mathematics is a very creative enterprise. They will also see that mathematics has an esthetic with certain agreed upon principles of elegance and beauty. For example, a set that is not closed under an operation may be considered “unsettling,” or a proof such as the one for the sum of a rational and an irrational may be simple and clear.

**Vocabulary**

Closed set

Closure of sets under an operation

Complex numbers

Complex conjugate

Imaginary numbers

Integers

Irrational numbers

Natural numbers (counting numbers)

Rational numbers

Real numbers

Sets of numbers

Whole numbers

**Resources and Materials**

**Activities 2.3.1, 2.3.2, 2.3.3, 2.3.4 should be completed in this investigation**. Stem-intending students should complete **Activity 2.3.3a** instead of **Activity 2.3.3**.

Activity 2.3.1 Closure and Sets of Numbers

Activity 2.3.1 Teacher Notes: Closure and Sets of Numbers

Activity 2.3.2 Imaginary Numbers

Activity 2.3.3 Complex Numbers

Activity 2.3.3a (+) Complex Numbers

Activity 2.3.4 Complex Solutions to Quadratic Equations

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