**Activity 2.3.1 Closure and Sets of Numbers**

TEACHER NOTES:

To help students remember the names of sets of numbers you can tell a stylized, fictional story about how various sets of numbers may have come about: perhaps a prehistoric family wanted to keep track of the animals it was domesticating. Perhaps the family kept their goats in a cave at night to protect them. Each night, they put one rock by the cave entrance to signify one goat. Each morning, they move the rocks to the other side of the entrance to signify the number of goats out in the pasture. The desire for more efficient methods motivated people to grunt or use syllables instead of rocks. Eventually the syllables became numbers. The natural (counting) numbers are born. Over time, they also develop the concept of 0. Zero together with the counting numbers are the Whole Numbers (the word “whole” has a oh or 0 in it) . As the family became more sophisticated and began to borrow and trade goats, they needed negative numbers. Whole numbers and their opposites are the integers. Perhaps the family is farming as well, and they are trading parts of a sack of grain for pottery, so they invent fractions. Rational numbers are the ratio of any two integers. Next, they desire to partition a square livestock pen along the diagonal. Using the Pythagorean theorem to find its length of the diagonal for the sides of 1 cubit long, gives them irrational numbers. The rationals together with the irrationals make up the reals.

Note: this timeline for the invention of numbers is historically **in**accurate, but it helps students recall which sets of numbers are subsets of others. Historically speaking, for example, irrational numbers were known to the Pythagoreans, but negative numbers did not appear until the Renaissance.

Have students write the definitions of the following sets of numbers:

 The need for negative numbers to close the whole numbers under subtraction gives us the integers Z –. Have students debate whether the integers are closed under each operation, and realizing that 3 divided by 6 is not an integer motivates the need for fractions (rational numbers Q – why is Q the symbol for the rational numbers?) Next tell the urban legend about the Pythagoreans’ horror of finding that the hypotenuse of a right triangle with legs = 1 unit is not a rational number. Thus the irrationals are discovered because the square root operation is not closed under the positive numbers . This is not to say that the closure of the positive rationals under the square root operator gives us all the irrational numbers. The irrationals are oh so much bigger than that If we consider all the square roots of rational numbers and the sum and product of these numbers with the rational, then we still do not get all the real numbers.

You may want use a number line to point out that rational (and irrational) numbers are dense: between any two rationals is another rational. Contrast this with the integers that are not dense ( there is no integer between -4 and -5) . Note that the reals are dense, that between any two real numbers is another real number and how the irrationals “plug holes” between the rationals. You can show the hole in the number line at $π, $with rational numbers like 3, 3.1, 3.14, 3.141 … approaching π$ $from below, and 4, 3.2, 3.15, 3.142…getting closer to $π $from above, but never being exactly π. A number line illustrates how irrationals fill in the gaps left by rationals. In fact, there are more irrational numbers than there are rationals. The rationals are “countably big” and the irrationals are uncountable.

The Activity includes questions about (a) the closure of sets, (b) the idea that irrational numbers can be understood as the number squeezed between ever more converging pairs of rational numbers. (e.g. 1.4<$\sqrt{2}$ <1.5 , (c) the characterization of irrationals in terms of non-repeating, non-terminating decimals and rationals as repeating decimals; (c) some surprising results such as .999…=1 , (d) questions about a decimal pattern not being the same as a repeating decimal e.g., .1012012301234… (e) common errors students make with numbers such as “fill in the blank with >,<, or = : .667\_?\_ 2/3 ; .3…. \_? \_.333; $π$\_\_?\_\_ 3.14.

The part of the activity that asks students to define the sets of numbers and to create a Venn Diagram for the types of real numbers could be assigned for homework .

The last page shows a proof by contradiction for the theorem “ The sum of a rational number and an irrational number is an irrational number.”  To send the message to students that that proof in algebra is just as important as in geometry, be sure to have students write the proof for the second theorem, even if all they do is modify the sample proof of the first theorem.